No Quantum Speedup over Gradient Descent

Lower Bounds for Convex Optimization

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The Gradient Descent method [Cauchy ‘47]

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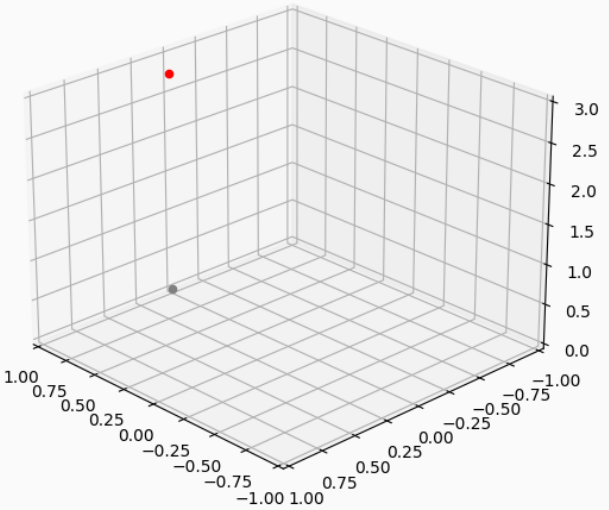

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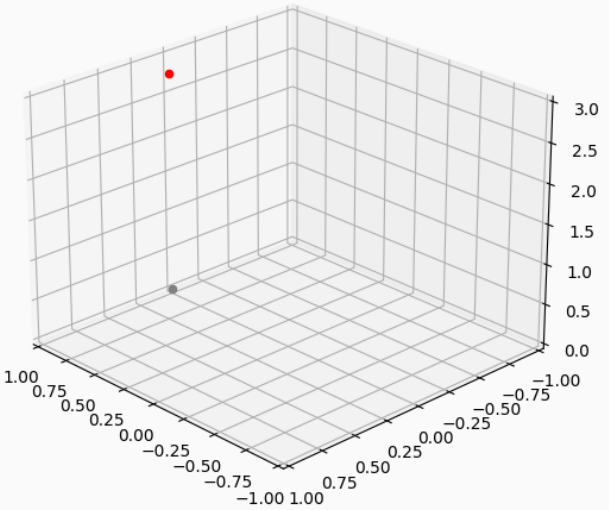

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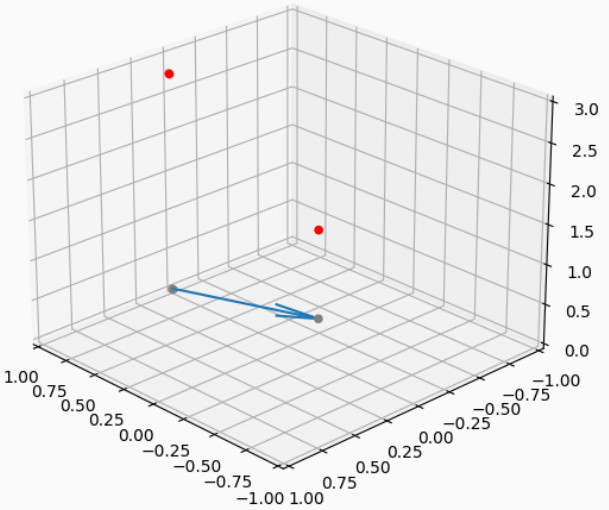

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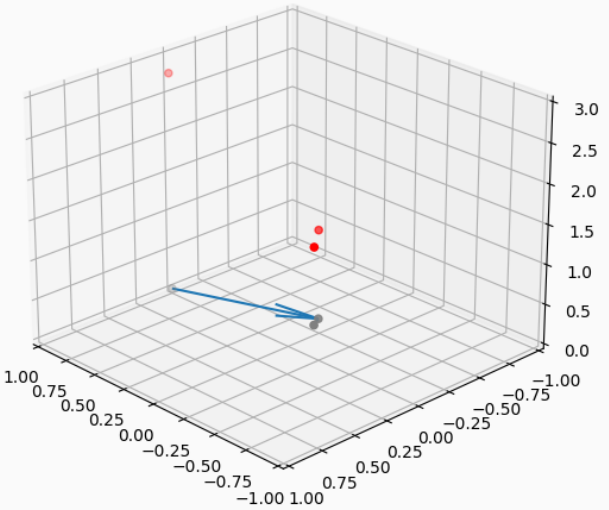

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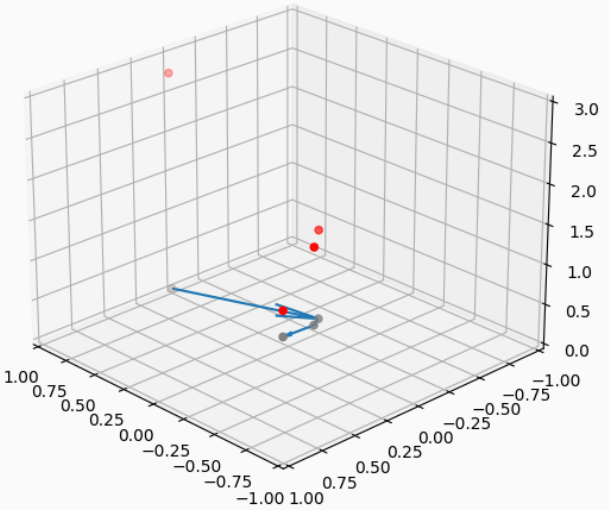

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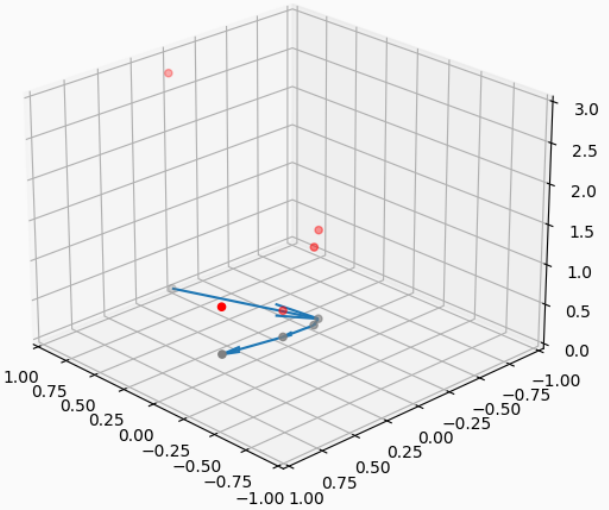

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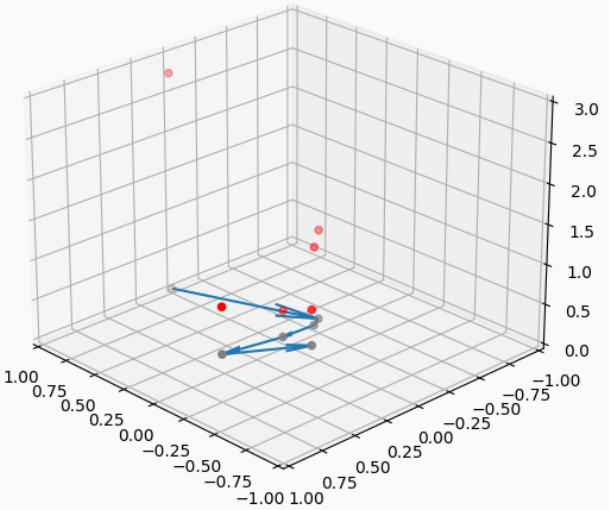

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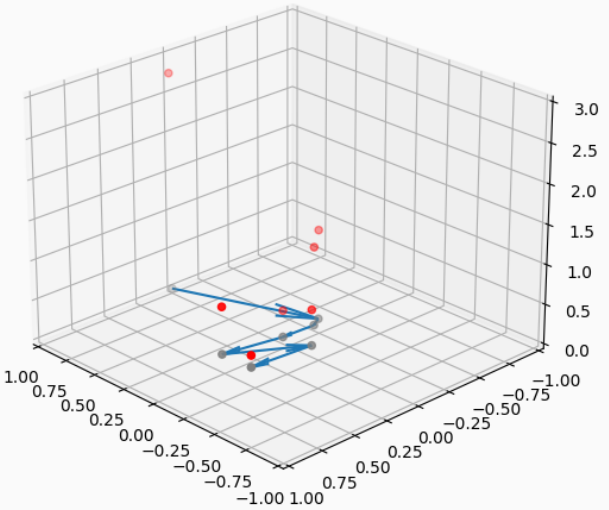

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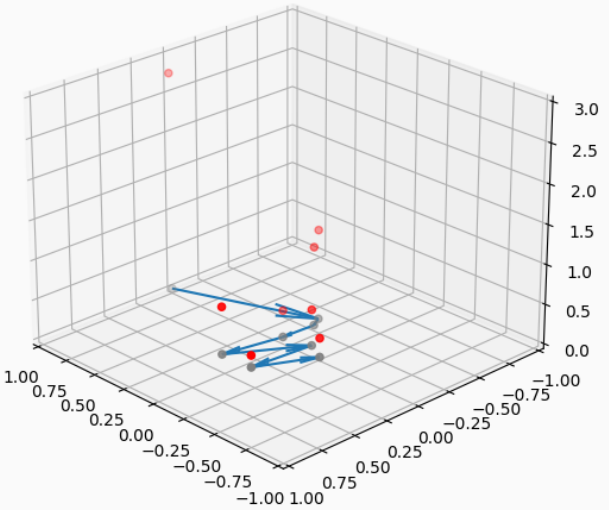

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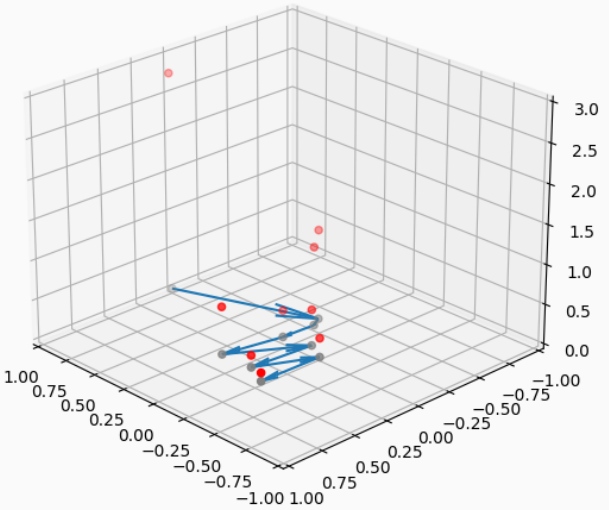

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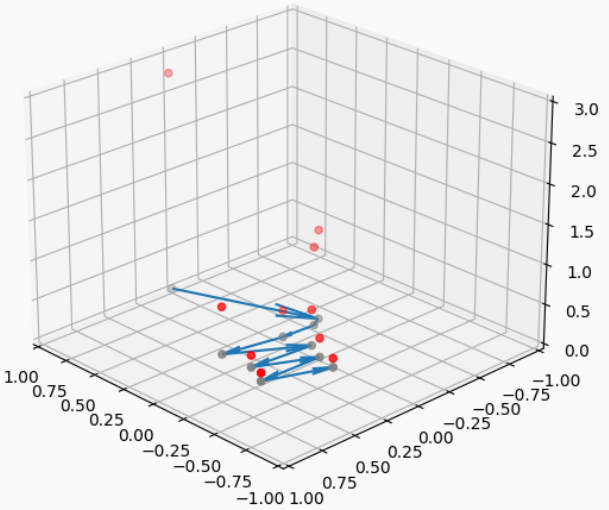

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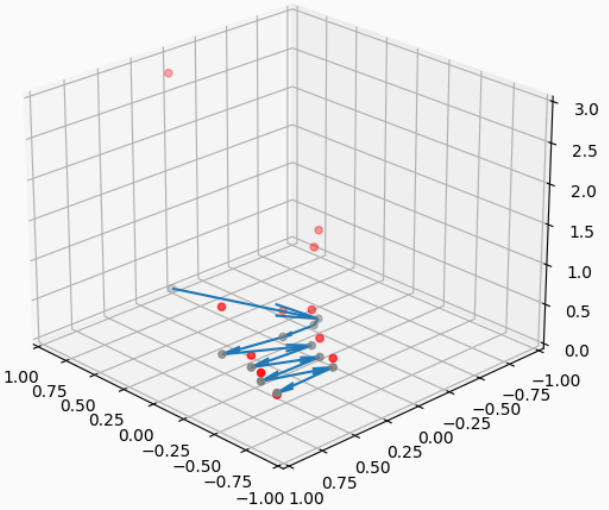

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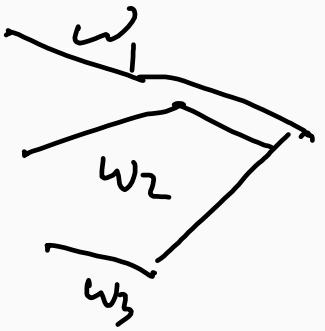

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  - Hard: Requires $\sqrt{\text{dimension}}$ many function evaluations.

$f(x) = \langle x, v \rangle$ Gradient $= v$, $\|v\| = 1$

Query x_1 $\Rightarrow$ Get $\langle x_1, v \rangle$ w.h.p. $\frac{1}{\sqrt{n}}$

(n dim)

Need $\sqrt{n}$ queries to learn v.

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- Guarantees?

First-Order Convex Optimization

The Task

Given: a convex region B, first-order oracle access to a convex function $f : \mathbb{R}^n \to \mathbb{R}$.

On input x, oracle O_f returns $f(x), \nabla f(x)$.

The Task

Given: a convex region B, first-order oracle access to a convex function $f : \mathbb{R}^n \to \mathbb{R}$.

$$\text{Find } x^* = \arg\min_{x \in B} f(x).$$

The Task

Given: a convex region B, first-order oracle access to a convex function $f : \mathbb{R}^n \to \mathbb{R}$.

Find $x' \in B$ s.t. $f(x') \leq \min_{x \in B} f(x) + \epsilon$.

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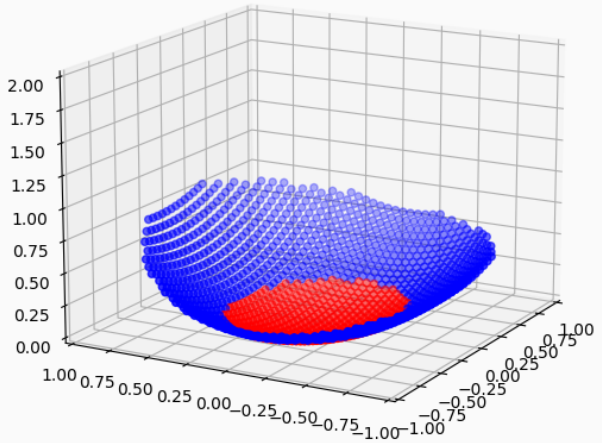

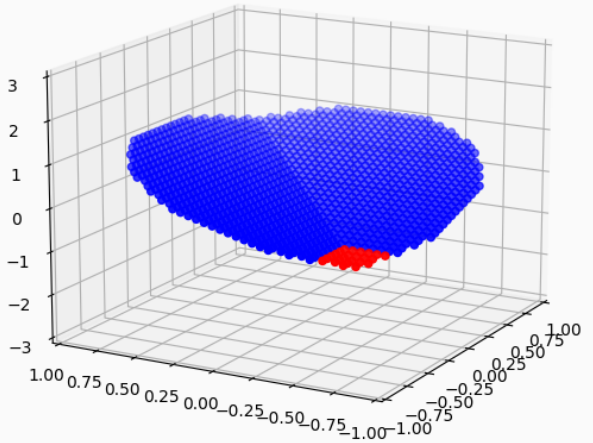

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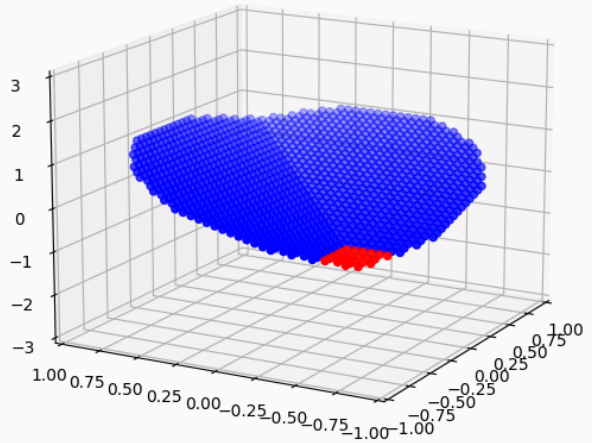
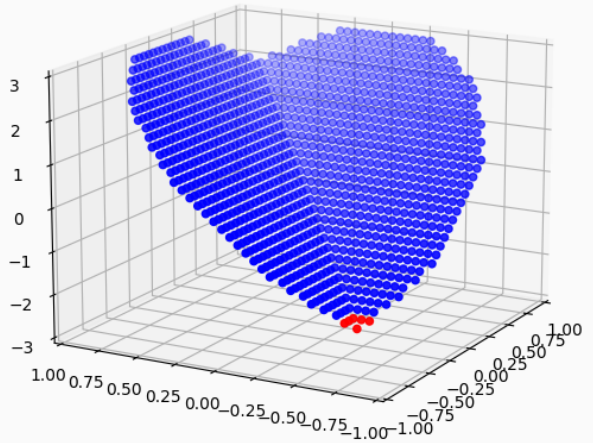


ϵ-opt



3ϵ-opt

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ϵ-optimal for G-Lipschitz function in ball of radius R

$\Longleftrightarrow$

ϵ/GR-optimal for 1-Lipschitz function in ball of radius 1

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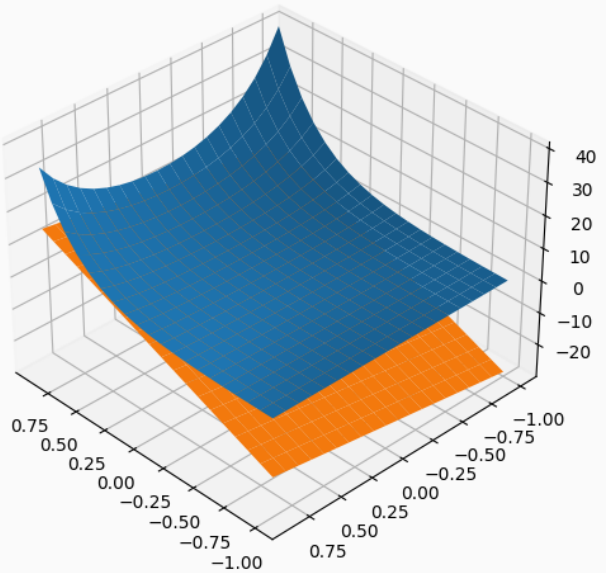

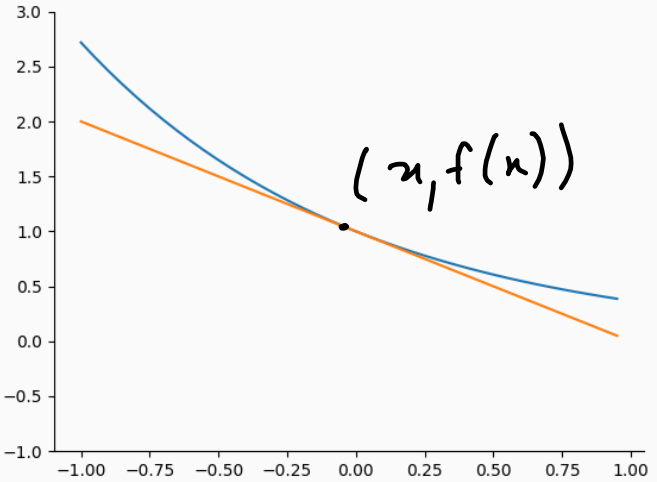

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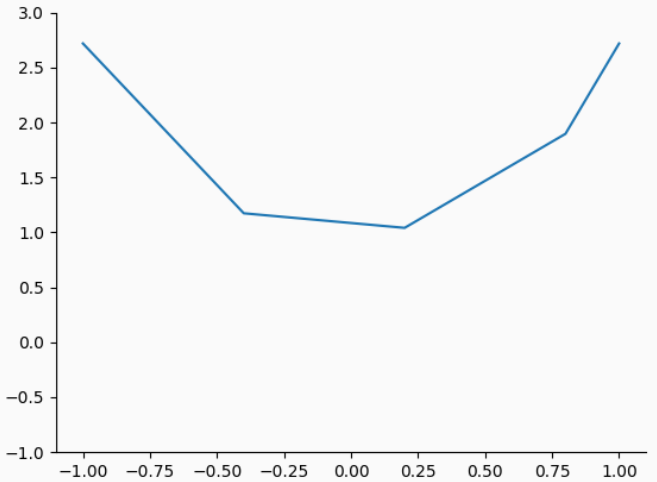

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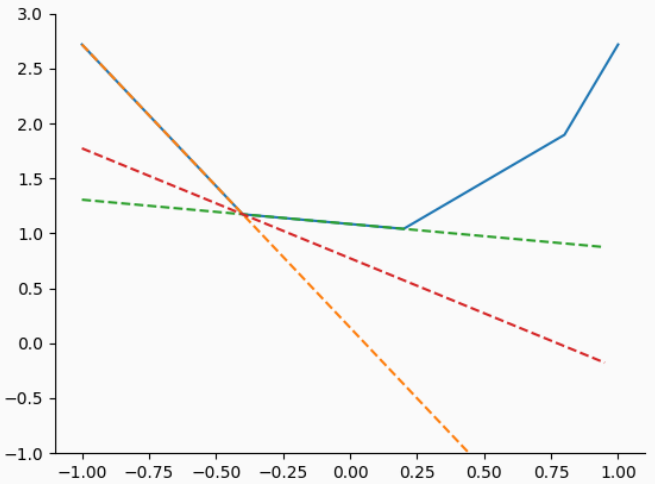

$g \in \nabla f(x) \Leftrightarrow f(x + v) \geq f(x) + \langle v, g \rangle$ for all v

Known Algorithms I

1-dimensional fn:

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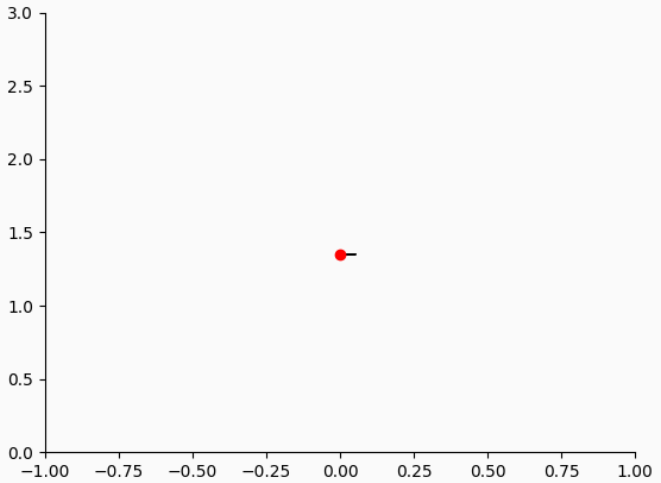

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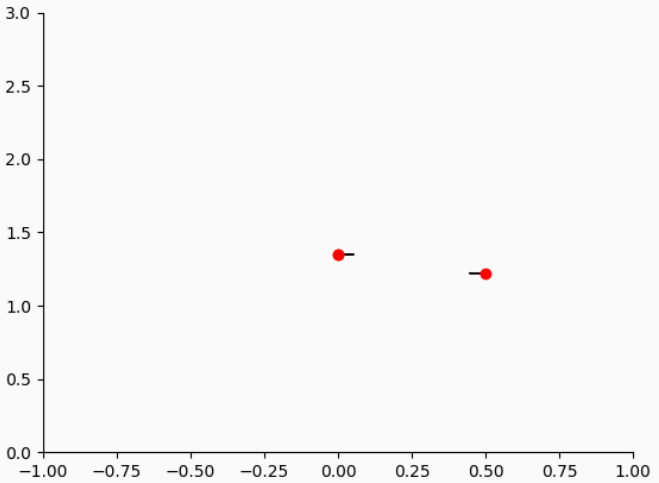

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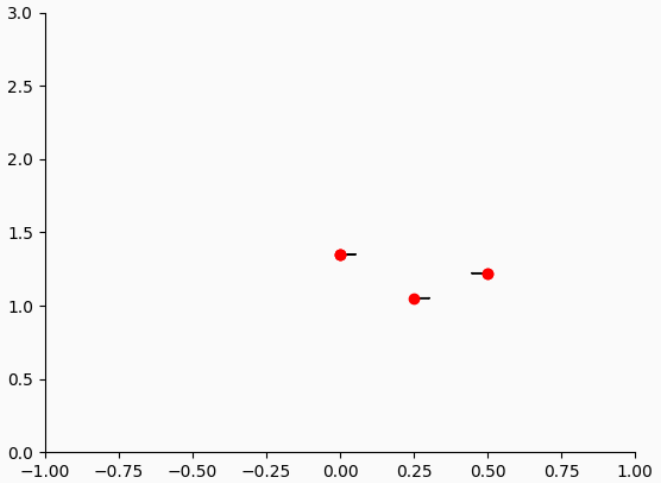

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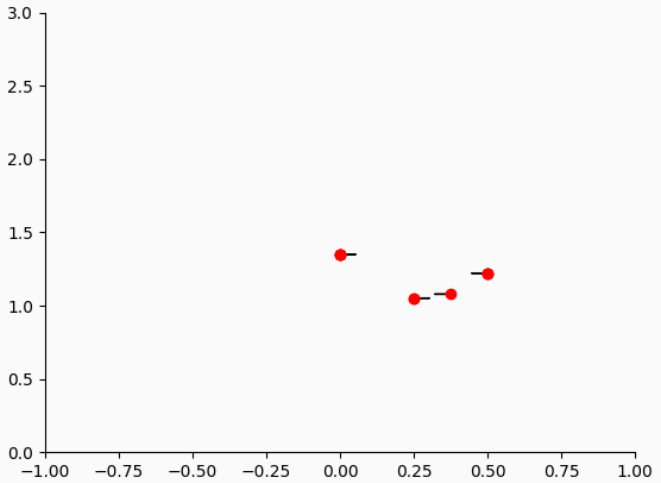

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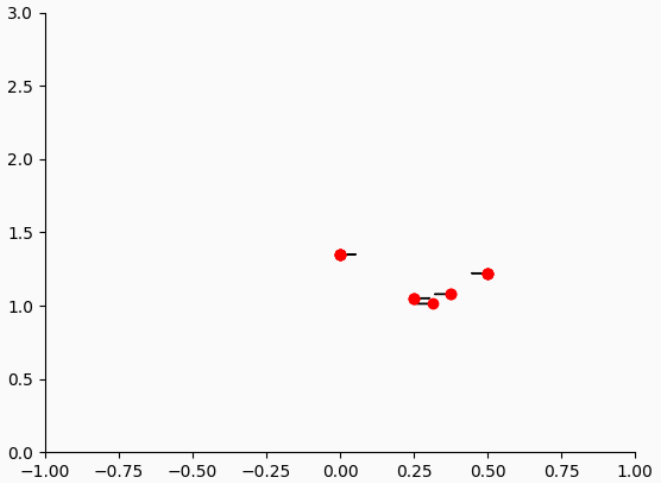

Known Algorithms I

1-dimensional fn:

$\log(1/\epsilon)$ steps.

Known Algorithms I

n-dimensional fn:

Known Algorithms I

n-dimensional fn:
(Center of Gravity Method)

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n-dimensional fn:
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$$\approx \log\left(\frac{Vol(B(1))}{Vol(B(\epsilon))}\right)$$

$$= n\log(1/\epsilon) \text{ steps.}$$

Known Algorithms II

Center of Gravity Method

$n \log(1/\epsilon)$ steps.

Projected Subgradient Descent

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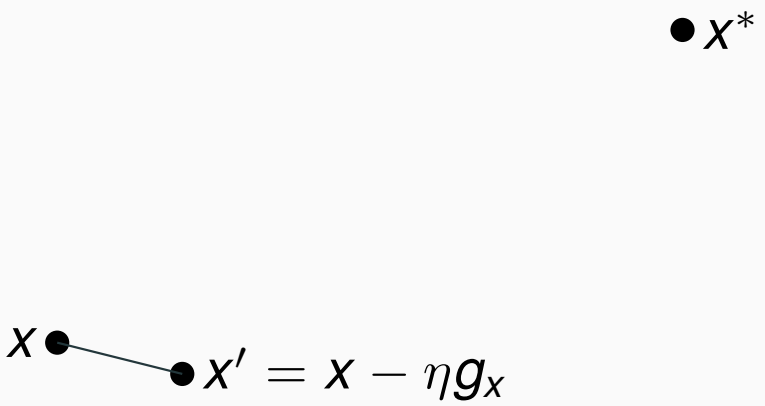

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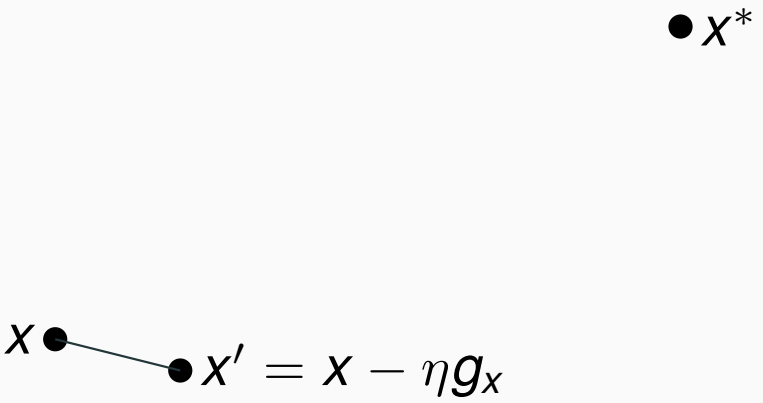

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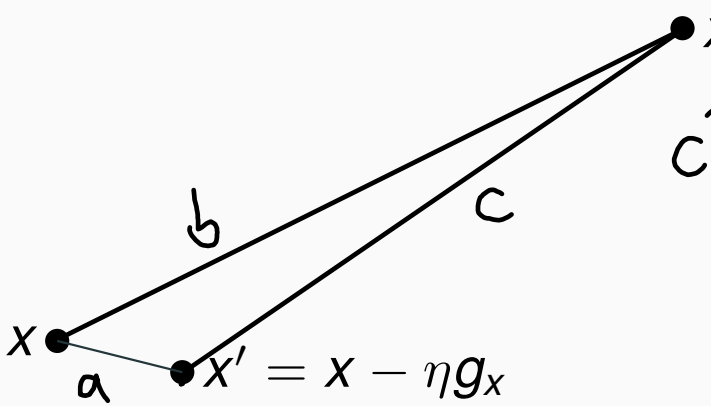

$$c^2 = a^2 + b^2 - 2\langle \vec{a}, \vec{b} \rangle$$

$$\langle -g_x, x^* - x \rangle \geq f(x) - f(x^*).$$

$$\|x^* - x'\|^2 \leq \|x^* - x\|^2 + \eta^2 - 2\eta(f(x) - f(x^*)).$$

Set $\eta = \epsilon$ .. If $f(x) - f(x^*) \geq \epsilon$, decr. by ϵ^2

$\therefore$ in $\frac{1}{\epsilon^2}$ steps, reach an ϵ-opt pt.

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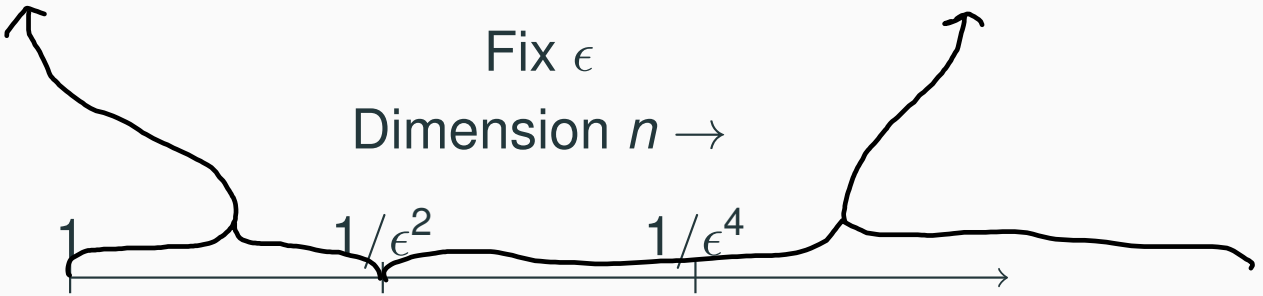

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$1/\epsilon^2$ steps.

Fix ϵ

Dimension $n \to$

Classical: Deterministic

Known Algorithms II

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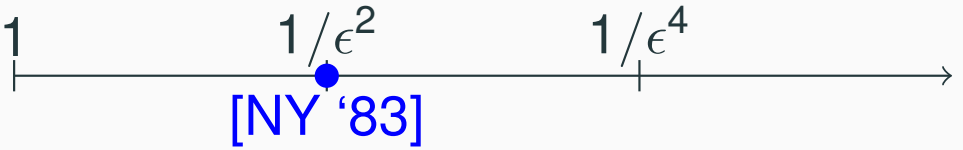

[NY '83]

$\Omega(1/\epsilon^2)$ queries

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[BJLLS '19]

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[BJLLS '19]

$\Omega(1/\epsilon^2)$ queries

Classical: Randomized

Theorem (Garg-Kothari-Netrapalli-S. '20)

For any $\epsilon > 0$, there is a family of 1-Lipschitz functions $\{f : \mathbb{R}^n \to \mathbb{R}\}$ with $n = \Theta(1/\epsilon^2)$ such that any randomized algorithm solving first-order convex optimization on these requires $\Omega(1/\epsilon^2)$ queries.

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- Find functions that encode information.
- ϵ-minimizing $f \implies$ Learning the encoded information.
- Each query should access the information in a controlled manner.

Lower Bounds

Randomized Lower Bound

The Base Function

$$f : \mathbb{R}^n \to \mathbb{R}$$

$$f(x) = \max\{x_1, x_2, \ldots, x_n\}.$$

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If x_i is a maximum, then e_i is a subgradient.

Function Class

$$z \in \{+1, -1\}^n$$

$$f_z(x) = \max\{z_1 x_1, z_2 x_2, \ldots, z_n x_n\}.$$

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$$\text{Set } \epsilon = \frac{.9}{\sqrt{n}}. \qquad x \text{ is } \epsilon\text{-opt} \Rightarrow f(x) \leq \frac{-.1}{\sqrt{n}}$$

Function Class

$$z \in \{+1, -1\}^n$$

$$f_z(x) = \max\{z_1 x_1, z_2 x_2, \ldots, z_n x_n\}.$$

$$\text{Minimum} = -\frac{1}{\sqrt{n}}, \text{ at } x = \left(-\frac{z_1}{\sqrt{n}}, \ldots, -\frac{z_n}{\sqrt{n}}\right).$$

$$\text{Set } \epsilon = \frac{.9}{\sqrt{n}}.$$

The behaviour of f

z_1 z_2 z_3 z_4 z_n

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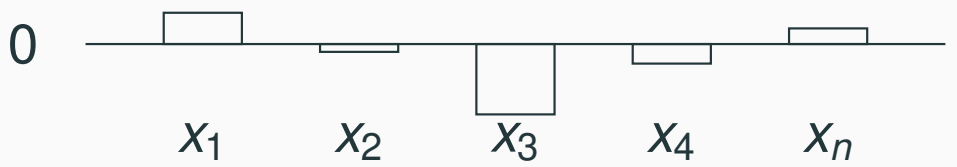

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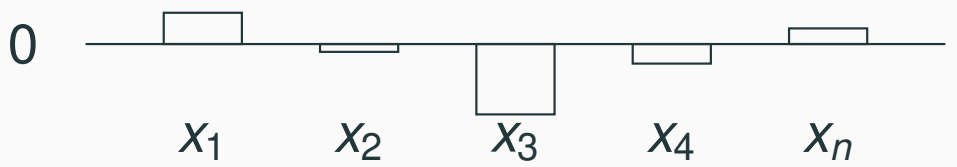
The behaviour of f

z_1 z_2 z_3 z_4 z_n

$+$



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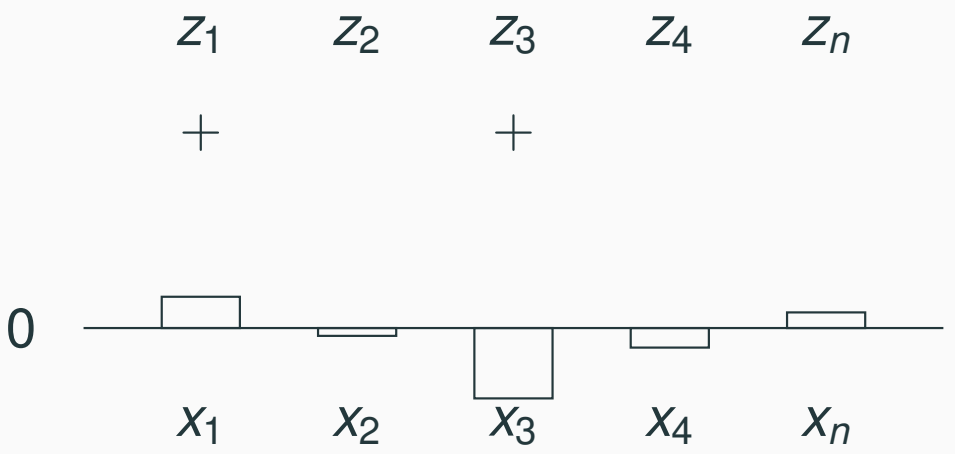

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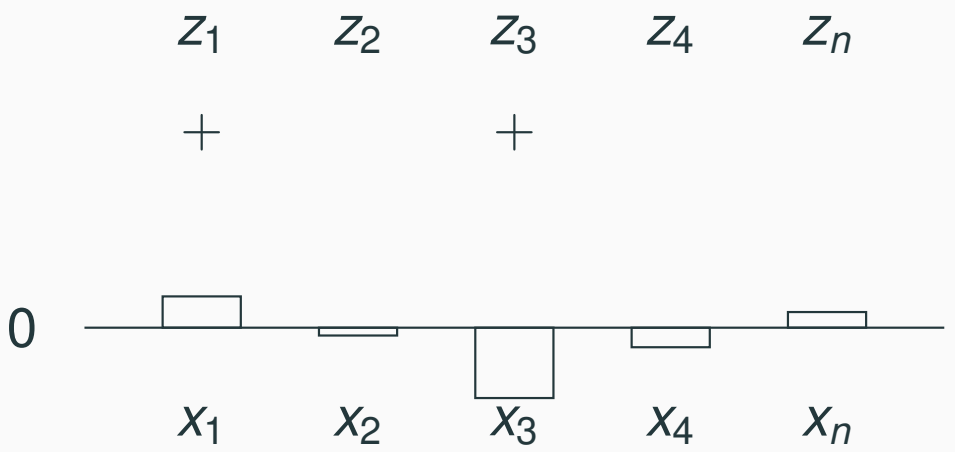

$\approx$ 2 bits of z revealed per query.

Function Class

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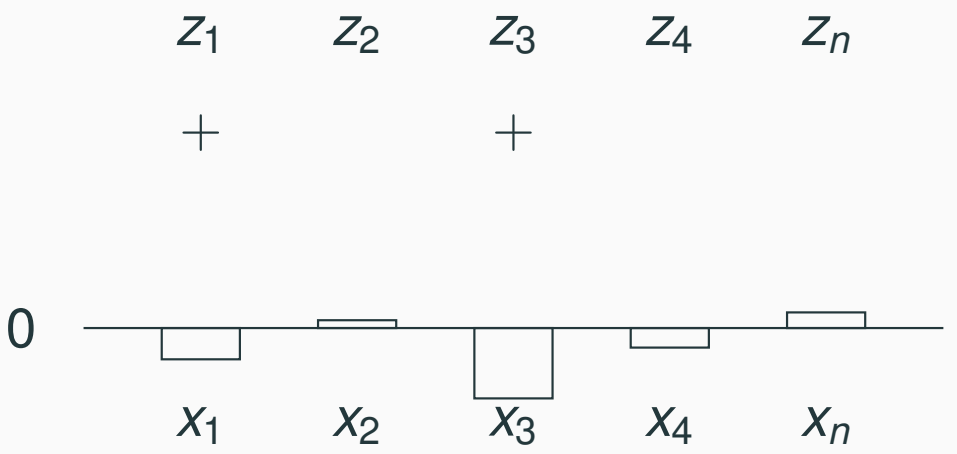
The behaviour of f



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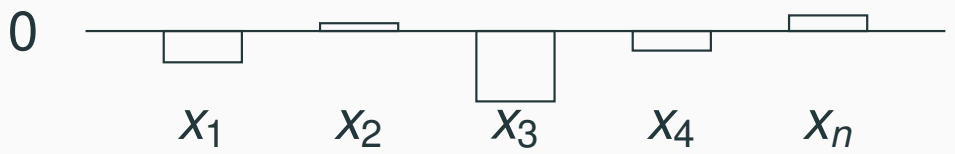
The behaviour of f

z_1	z_2	z_3	z_4	z_n
$+$	$-$	$+$	$+$	$-$



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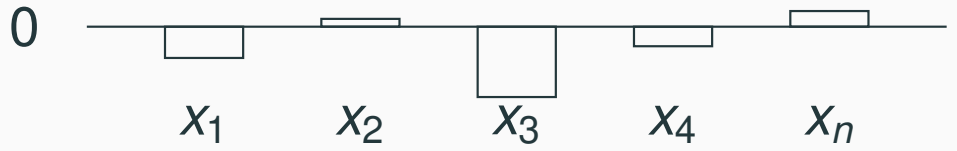

Finding ϵ-optimal point $\implies$ learning z.

Need $\Omega(n)$ queries $\left(\Omega\left(\frac{1}{\epsilon^2}\right) \text{ queries}\right)$

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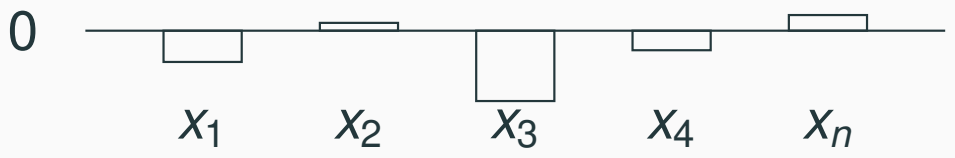
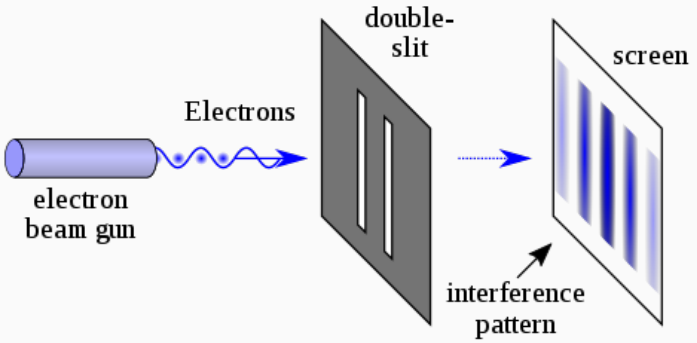


queries.

Requires $\Omega(n) = \Omega(1/\epsilon^2)$

Quantum Computing

Quantum Reality



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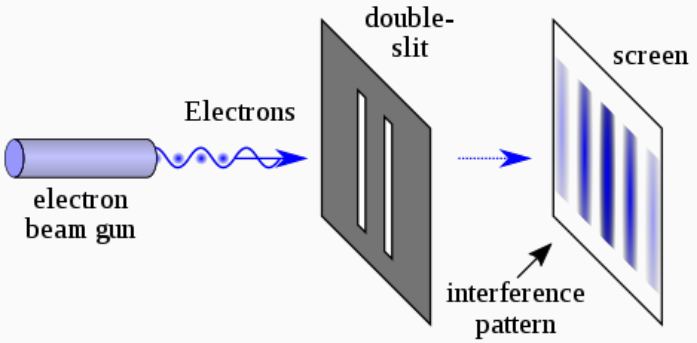
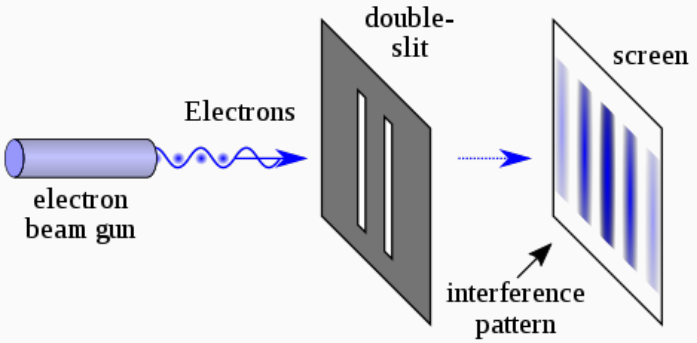


Particles can be in a superposition of states.

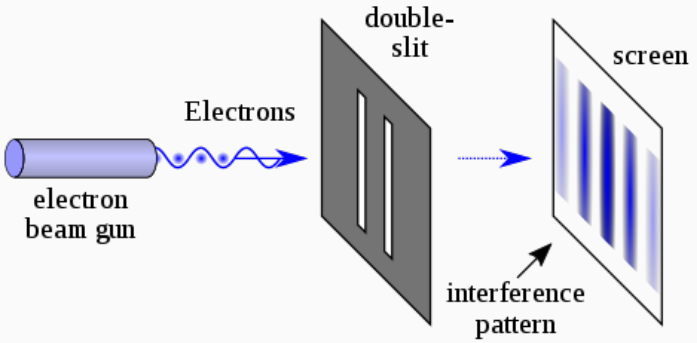
Quantum Reality



Particles can be in a superposition of states.

$$\alpha|\text{Left Slit}\rangle + \beta|\text{Right Slit}\rangle, \text{ with } |\alpha|^2 + |\beta|^2 = 1.$$

Quantum Reality



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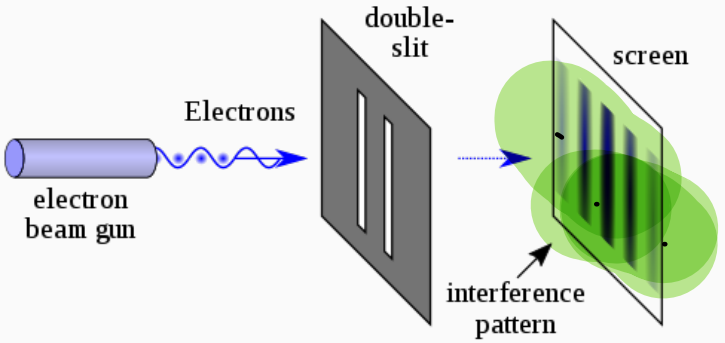

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$$|\text{Left Slit}\rangle \mapsto |\text{Left Spread}\rangle \text{ and } |\text{Right Slit}\rangle \mapsto |\text{Right Spread}\rangle.$$

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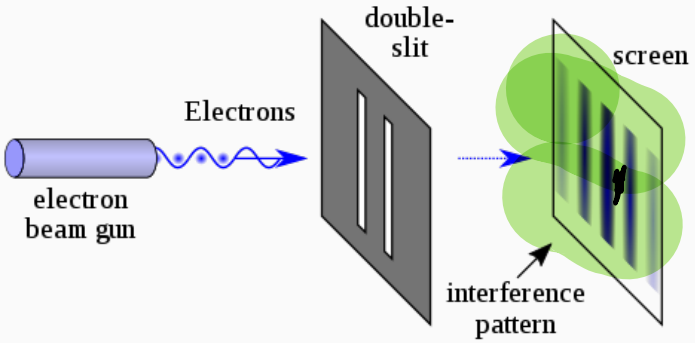

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$$|\text{Left Slit}\rangle \mapsto |\text{Left Spread}\rangle \text{ and } |\text{Right Slit}\rangle \mapsto |\text{Right Spread}\rangle.$$

The actual state is the sum of the states of each component:
$\alpha|\text{Left Spread}\rangle + \beta|\text{Right Spread}\rangle$.

Quantum Computer

- A classical computer has a memory that is in one of $2^{\text{memory size in bits}}$ states.

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$$|\phi\rangle_{MEMORY} = \sum_{m \in \{0,1\}^{\text{memory size}}} \alpha_m |m\rangle_{MEMORY} \text{ with } \sum |\alpha_m|^2 = 1.$$

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- Permitted operations are those that preserve norms.
- Can apply a unitary M to the state $|\phi\rangle$ to get the new state $|M\phi\rangle$.
  - Every base state gets mapped to a valid superposition.

$$|m\rangle \longmapsto |\phi_m\rangle$$

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- Permitted operations are those that preserve norms.
- Can apply a unitary M to the state $|\phi\rangle$ to get the new state $|M\phi\rangle$.
  - Every base state gets mapped to a valid superposition.
  - Each of these mappings are orthonormal.

$|\phi_{m_1}\rangle$

$|\phi_{m_2}\rangle$

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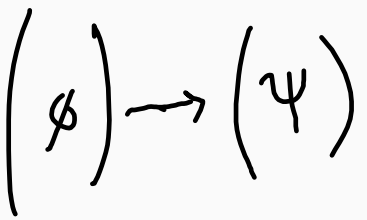
- Permitted operations are those that preserve norms.
- Can apply a unitary M to the state $|\phi\rangle$ to get the new state $|M\phi\rangle$.
  - Every base state gets mapped to a valid superposition.
  - Each of these mappings are orthonormal.
  - Just changing the basis.

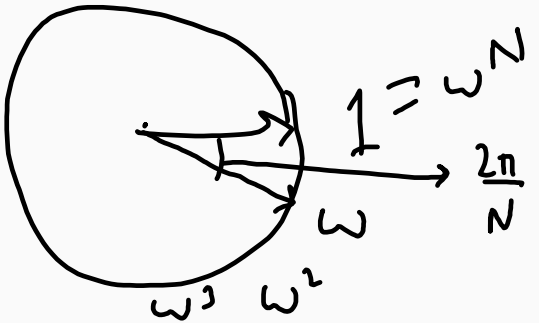


Quantum Tricks

- Can take advantage of Fourier transforms.
- The vectors

$$\frac{1}{\sqrt{N}}\begin{pmatrix}1\\1\\1\\\vdots\\1\end{pmatrix}, \frac{1}{\sqrt{N}}\begin{pmatrix}1\\\omega\\\omega^2\\\vdots\\\omega^{N-1}\end{pmatrix}, \frac{1}{\sqrt{N}}\begin{pmatrix}1\\\omega^2\\\omega^4\\\vdots\\\omega^{2(N-1)}\end{pmatrix}, \frac{1}{\sqrt{N}}\begin{pmatrix}1\\\omega^3\\\omega^6\\\vdots\\\omega^{3(N-1)}\end{pmatrix}, \ldots$$

are all orthonormal where $\omega = e^{i2\pi/N}$ is the principal Nth root of unity.



Computing Gradients

$$f : \mathbb{R}^2 \to \mathbb{R}$$

Computing Gradients

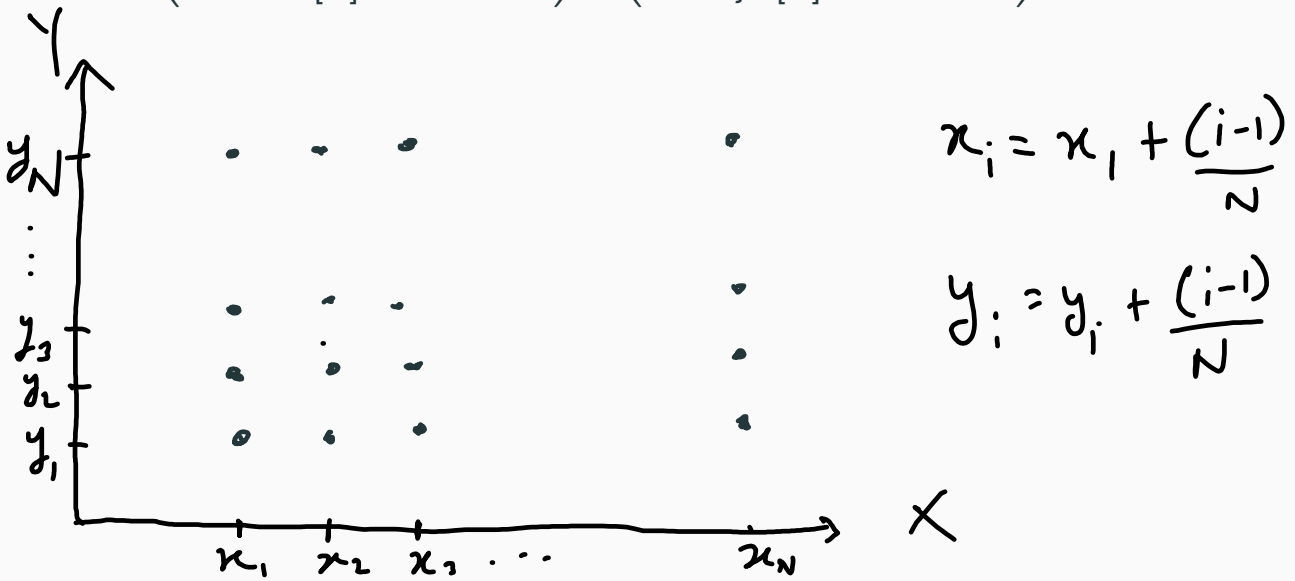
$$f : \mathbb{R}^2 \to \mathbb{R}$$

Example state: $\left(\sum_i \alpha_i |x_i y_i\rangle_{INPUT} \right) |0\rangle_{OUTPUT} |0\rangle_{OTHER\_VARS}$

Computing Gradients

$$f : \mathbb{R}^2 \to \mathbb{R},\ f(x, y) = ax + by$$

Initial state: $\left(\frac{1}{\sqrt{N}} \sum_{i \in [N]} |x_i\rangle_{INPUT_1}\right) \left(\frac{1}{\sqrt{N}} \sum_{j \in [N]} |y_j\rangle_{INPUT_2}\right) |0\rangle_{OUTPUT}$



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Initial state: $$\left(\frac{1}{\sqrt{N}} \sum_{i \in [N]} |x_i\rangle_{INPUT_1} \right) \left(\frac{1}{\sqrt{N}} \sum_{j \in [N]} |y_j\rangle_{INPUT_2} \right) |0\rangle_{OUTPUT}$$

Query: $$\frac{1}{N} \sum_{i \in [N]} \sum_{j \in [N]} |x_i\rangle_{INPUT_1} |y_j\rangle_{INPUT_2} |f(x_i, y_j)\rangle_{OUTPUT}$$

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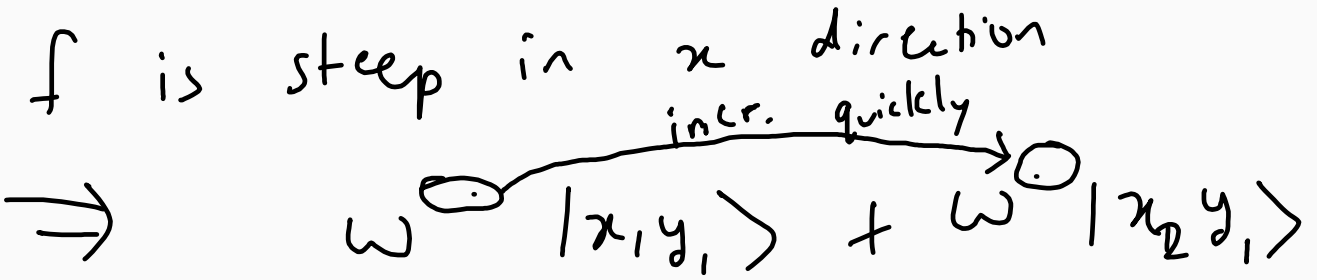
Initial state: $\left(\frac{1}{\sqrt{N}} \sum_{i \in [N]} |x_i\rangle_{INPUT_1}\right) \left(\frac{1}{\sqrt{N}} \sum_{j \in [N]} |y_j\rangle_{INPUT_2}\right) |0\rangle_{OUTPUT}$

Query: $\frac{1}{N} \sum_{i \in [N]} \sum_{j \in [N]} |x_i\rangle_{INPUT_1} |y_j\rangle_{INPUT_2} |f(x_i, y_j)\rangle_{OUTPUT}$

Add phases: $\frac{1}{N} \sum_{i \in [N]} \sum_{j \in [N]} \omega^{\frac{f(x_i, y_j) N^2}{3}} |x_i\rangle_{INPUT_1} |y_j\rangle_{INPUT_2} |f(x_i, y_j)\rangle_{OUTPUT}$



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$$f : \mathbb{R}^2 \to \mathbb{R},\ f(x, y) = ax + by$$

Initial state: $\left(\frac{1}{\sqrt{N}} \sum_{i \in [N]} |x_i\rangle_{INPUT_1}\right) \left(\frac{1}{\sqrt{N}} \sum_{j \in [N]} |y_j\rangle_{INPUT_2}\right) |0\rangle_{OUTPUT}$

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Query again (to undo): $\frac{1}{N} \sum_{i \in [N]} \sum_{j \in [N]} \omega^{\frac{f(x_i, y_j) N^2}{3}} |x_i\rangle_{INPUT_1} |y_j\rangle_{INPUT_2} |0\rangle_{OUTPUT}$

Needs query b/c $|x_1 y_1\rangle|c_1\rangle \mapsto |x_1 y_1\rangle|0\rangle$, $|x_1 y_1\rangle|c_2\rangle \mapsto |x_1 y_1\rangle|0\rangle$ is not unitary.

Only $|x_1 y_1\rangle|f(x_1, y_1)\rangle$ should map to $|x_1 y_1\rangle|0\rangle$

See slide 15 for why query undoes.

Computing Gradients

$$f : \mathbb{R}^2 \to \mathbb{R},\ f(x, y) = ax + by$$

Initial state: $\left(\frac{1}{\sqrt{N}} \sum_{i \in [N]} |x_i\rangle_{INPUT_1}\right) \left(\frac{1}{\sqrt{N}} \sum_{j \in [N]} |y_j\rangle_{INPUT_2}\right) |0\rangle_{OUTPUT}$

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Query again (to undo): $\frac{1}{N} \sum_{i \in [N]} \sum_{j \in [N]} \omega^{\frac{f(x_i, y_j) N^2}{3}} |x_i\rangle_{INPUT_1} |y_j\rangle_{INPUT_2} |0\rangle_{OUTPUT}$

Equals:

$$\omega^{(ax_1 + by_1)\frac{N^2}{3}} \left(\frac{1}{\sqrt{N}} \sum_{i \in [N]} \omega^{\frac{aiN}{3}} |x_1 + i/N\rangle_{INPUT_1}\right) \left(\frac{1}{\sqrt{N}} \sum_{j \in [N]} \omega^{\frac{bjN}{3}} |y_1 + j/N\rangle_{INPUT_2}\right) |0\rangle_{OUTPUT}$$

$\{0, 1, \ldots, N-1\}$ $\{0, 1, \ldots, N-1\}$

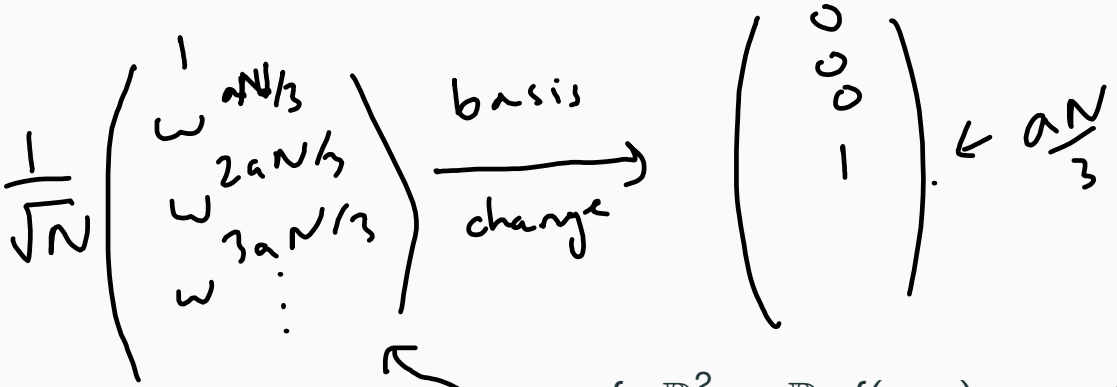
Computing Gradients

$$f : \mathbb{R}^2 \to \mathbb{R},\ f(x, y) = ax + by$$

$$\omega^{(ax_1+by_1)\frac{N^2}{3}} \left(\frac{1}{\sqrt{N}} \sum_{i \in [N]} \omega^{\frac{aiN}{3}} |x_1 + i/N\rangle_{INPUT_1} \right) \left(\frac{1}{\sqrt{N}} \sum_{j \in [N]} \omega^{\frac{bjN}{3}} |y_1 + j/N\rangle_{INPUT_2} \right) |0\rangle_{OUTPUT}$$

$\{0, \ldots, N-1\}$ $\{0, \ldots, N-1\}$

Computing Gradients



$$f : \mathbb{R}^2 \to \mathbb{R},\ f(x, y) = ax + by$$

$$\omega^{(ax_1+by_1)\frac{N^2}{3}} \left(\frac{1}{\sqrt{N}} \sum_{i \in [N]} \omega^{\frac{aiN}{3}} |x_1 + i/N\rangle_{INPUT_1} \right) \left(\frac{1}{\sqrt{N}} \sum_{j \in [N]} \omega^{\frac{bjN}{3}} |y_1 + j/N\rangle_{INPUT_2} \right) |0\rangle_{OUTPUT}$$

Change basis: $\omega^{f(x_1,y_1)\frac{N^2}{3}} |aN/3\rangle_{PHASE_1} \left(\frac{1}{\sqrt{N}} \sum_{j \in [N]} \omega^{\frac{bjN}{3}} |y_1 + j/N\rangle_{INPUT_2} \right) |0\rangle_{OUTPUT}$

Computing Gradients

$f : \mathbb{R}^n \to \mathbb{R}$ 2 queries

$$f : \mathbb{R}^2 \to \mathbb{R},\ f(x, y) = ax + by$$

$$\omega^{(ax_1+by_1)\frac{N^2}{3}} \left(\frac{1}{\sqrt{N}} \sum_{i\in[N]} \omega^{\frac{aiN}{3}} |x_1 + i/N\rangle_{INPUT_1} \right) \left(\frac{1}{\sqrt{N}} \sum_{j\in[N]} \omega^{\frac{bjN}{3}} |y_1 + j/N\rangle_{INPUT_2} \right) |0\rangle_{OUTPUT}$$

Change basis: $\omega^{f(x_1,y_1)\frac{N^2}{3}} |aN/3\rangle_{PHASE_1} \left(\frac{1}{\sqrt{N}} \sum_{j\in[N]} \omega^{\frac{bjN}{3}} |y_1 + j/N\rangle_{INPUT_2} \right) |0\rangle_{OUTPUT}$

For coord 2: $\omega^{f(x_1,y_1)\frac{N^2}{3}} |aN/3\rangle_{PHASE_1} |bN/3\rangle_{PHASE_2} |0\rangle_{OUTPUT}$

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Use First-Order Convex Optimization as a proxy for Gradient Descent.

Quantum Speedup

$$z \in \{+1, -1\}^n$$

$$f_z(x) = \max\{z_1 x_1, z_2 x_2, \ldots, z_n x_n\}.$$

$$\text{Minimum} = -\frac{1}{\sqrt{n}}, \text{ at } x = \left(-\frac{z_1}{\sqrt{n}}, \ldots, -\frac{z_n}{\sqrt{n}}\right).$$

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Can then use Belovs' algorithm to learn z from such OR queries.

Known Algorithms II

Center of Gravity Method

$n \log(1/\epsilon)$ steps.

Projected Subgradient Descent

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Dimension $n \to$

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[CCLW '18]

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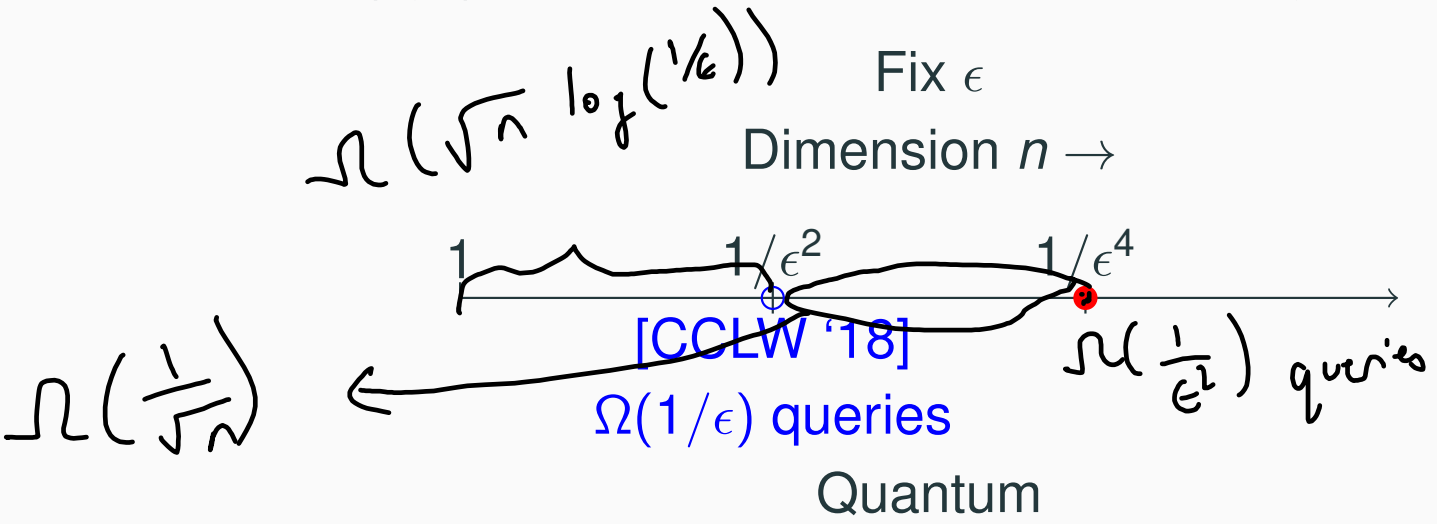

Theorem (Garg-Kothari-Netrapalli-S. '20)

For any $\epsilon > 0$, there is a family of 1-Lipschitz functions $\{f : \mathbb{R}^n \to \mathbb{R}\}$ with $n = \tilde{\Theta}(1/\epsilon^4)$ such that any quantum algorithm solving first-order convex optimization on these requires $\Omega(1/\epsilon^2)$ queries.

Quantum Computing

Quantum Lower Bound

The Query Model

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- For f, f' s.t. $f(x) = f'(x)$ and $\nabla f(x) = \nabla f'(x)$:

$$O_f |x\rangle_{INPUT} |\phi\rangle_{REST} = O_{f'} |x\rangle_{INPUT} |\phi\rangle_{REST}$$

The Base Function

"Complexity of Highly Parallel Non-Smooth Convex Optimization"

- Sébastien Bubeck, Qijia Jiang, Yin Tat Lee, Yuanzhi Li, Aaron Sidford

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$$f : \mathbb{R}^n \to \mathbb{R}$$

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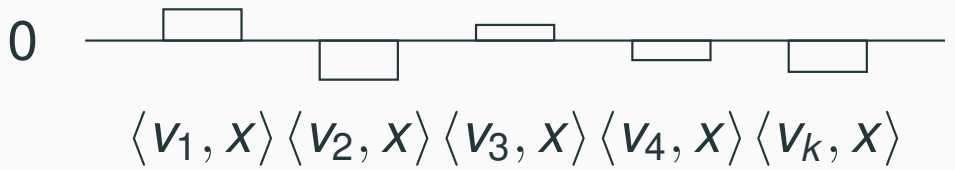

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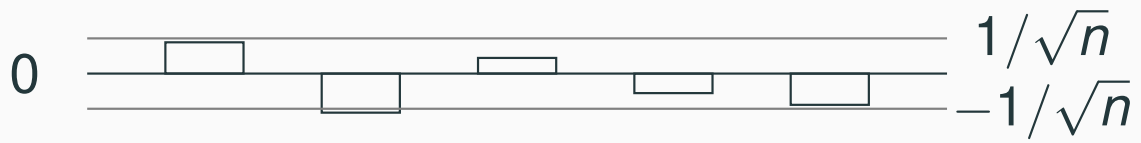

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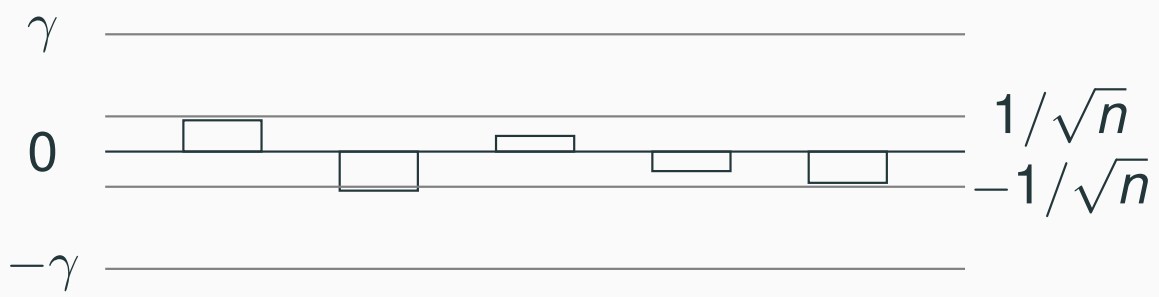

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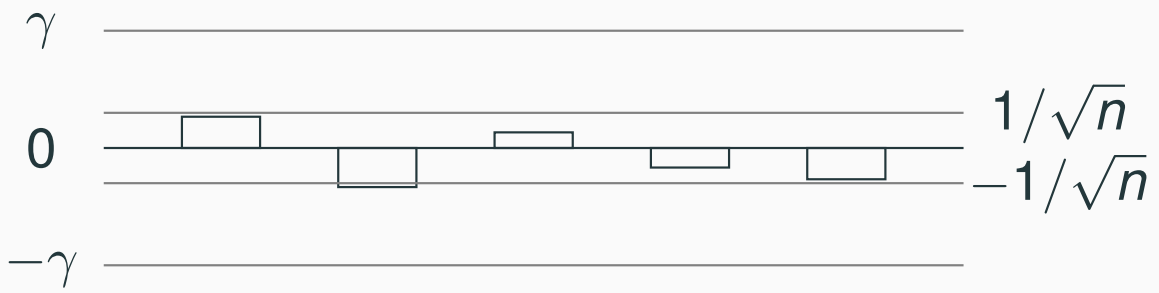

$v_2 \notin \{v \mid \langle v, x\rangle > 2\gamma\}$

Whp, first query reveals v_1.

v_2 through v_k nearly random from $n-1$ dimensional space.

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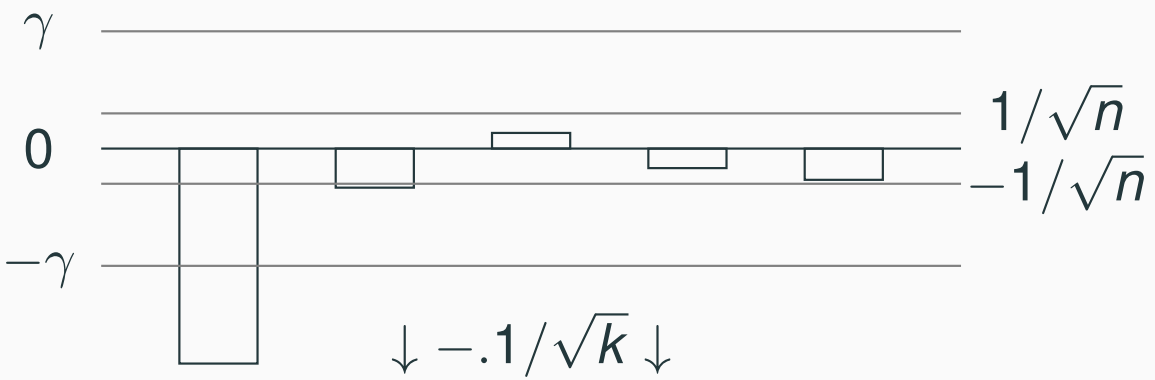

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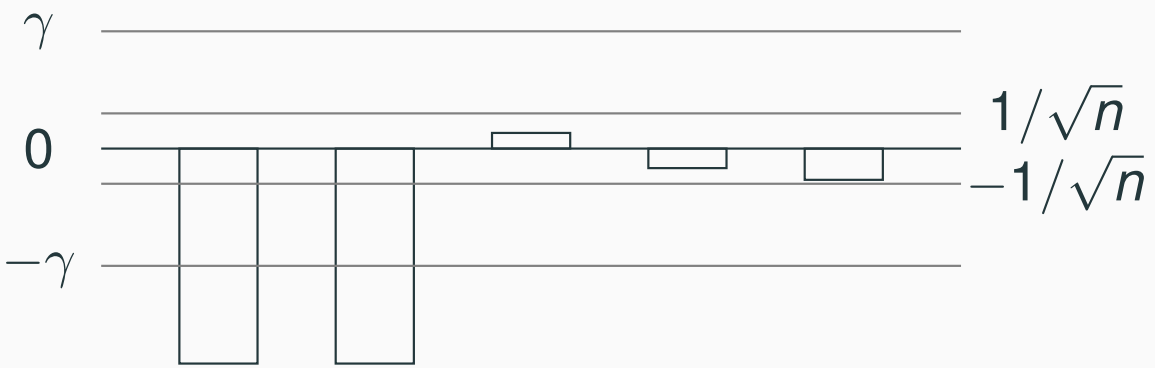

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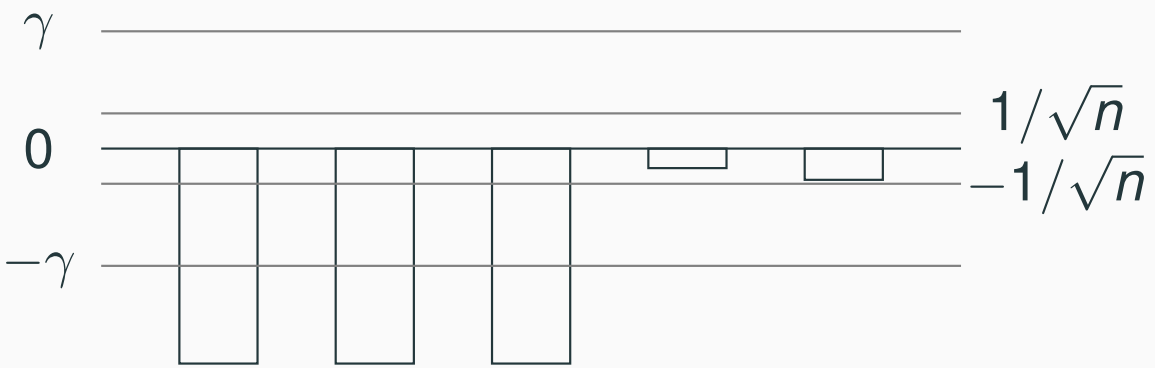

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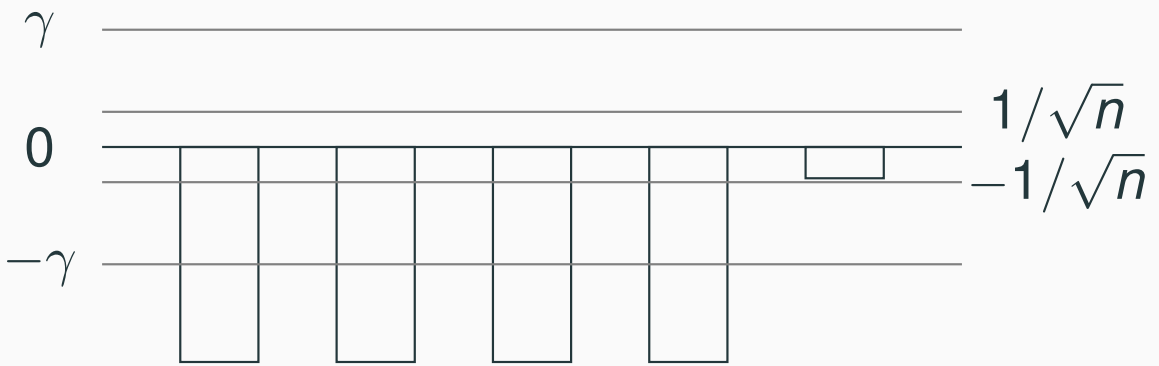

v_k still nearly at random from $n - k$ dimensional space.

Can't output ϵ-optimal point.

The Hybrid Argument

First query

$$
\begin{aligned}
&|x_1\rangle|\phi_1\rangle \\
+&|x_2\rangle|\phi_2\rangle \\
+&|x_3\rangle|\phi_3\rangle \\
+&|x_4\rangle|\phi_4\rangle \\
+&|x_5\rangle|\phi_5\rangle \\
+&\cdots
\end{aligned}
$$

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+&\cdots
\end{aligned}
$$

$\xrightarrow{\text{pass through oracle for } f_V(x)}$

First answer

$$
\begin{aligned}
&|x_1\rangle|\psi_1\rangle \\
+&|x_2\rangle|\psi_2\rangle \\
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+&|x_4\rangle|\psi_4\rangle \\
+&|x_5\rangle|\psi_5\rangle \\
+&\cdots
\end{aligned}
$$

The Hybrid Argument

First query

$$|x_1\rangle|\phi_1\rangle + |x_2\rangle|\phi_2\rangle + |x_3\rangle|\phi_3\rangle + |x_4\rangle|\phi_4\rangle + |x_5\rangle|\phi_5\rangle + \cdots$$

pass through oracle for $f_{(v_1)}(x)=\langle v_1, x\rangle$

w.h.p $f_v(x) = \langle v, x\rangle$

Corrupted answer

$$|x_1\rangle|\psi_1\rangle + |x_2\rangle|\psi_2\rangle + |x_3\rangle|\psi_3\rangle + |x_4\rangle|\psi_4'\rangle + |x_5\rangle|\psi_5\rangle + \cdots$$

The Hybrid Argument

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+&\cdots
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+&|x_4\rangle|\psi_4'\rangle \\
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+&\cdots
\end{aligned}
$$

Corrupted answer

- Changing oracle #1 barely changes the resulting state after 1 query. (with high probability)

The Hybrid Argument

First query

$$
\begin{aligned}
&|x_1\rangle|\phi_1\rangle \\
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$$

- Changing oracle #1 barely changes the resulting state after 1 query. (with high probability)
- The actual, corrupted states at the end are also close. (because quantum only does unitaries [which preserve distances])

The Hybrid Argument

First query

$$
\begin{aligned}
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\end{aligned}
$$

Corrupted answer

- Changing oracle #1 barely changes the resulting state after 1 query. (with high probability)
- The actual, corrupted states at the end are also close.
- Actual, corrupted algorithm nearly the same.

The Hybrid Argument: Second query

Second query (corrupted)

$$
\begin{aligned}
&|x_1\rangle|\tau_1\rangle \\
+&|x_2\rangle|\tau_2\rangle \\
+&|x_3\rangle|\tau_3\rangle \\
+&|x_4\rangle|\tau_4\rangle \\
+&|x_5\rangle|\tau_5\rangle \\
+&\cdots
\end{aligned}
$$

.

The Hybrid Argument: Second query

Second query (corrupted)

$$
\begin{aligned}
&|x_1\rangle|\tau_1\rangle \\
+&|x_2\rangle|\tau_2\rangle \\
+&|x_3\rangle|\tau_3\rangle \\
+&|x_4\rangle|\tau_4\rangle \\
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\end{aligned}
\xrightarrow{\text{pass through oracle for } f_V(x)}
$$

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\end{aligned}
$$

$\xrightarrow{\text{pass through oracle for } f_V(x)}$

Second answer (corrupted)

$$
\begin{aligned}
&|x_1\rangle|\chi_1\rangle \\
+&|x_2\rangle|\chi_2\rangle \\
+&|x_3\rangle|\chi_3\rangle \\
+&|x_4\rangle|\chi_4\rangle \\
+&|x_5\rangle|\chi_5\rangle \\
+&\cdots
\end{aligned}
$$

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Second query (corrupted)

$$|x_1\rangle|\tau_1\rangle + |x_2\rangle|\tau_2\rangle + |x_3\rangle|\tau_3\rangle + |x_4\rangle|\tau_4\rangle + |x_5\rangle|\tau_5\rangle + \cdots$$

pass through oracle for $f_{(v_1,v_2)}(x)=\max\{\langle v_1,x\rangle,\langle v_2,x\rangle-\gamma\}$

Second answer (twice corrupted)

$$|x_1\rangle|\chi_1\rangle + |x_2\rangle|\chi_2'\rangle + |x_3\rangle|\chi_3\rangle + |x_4\rangle|\chi_4\rangle + |x_5\rangle|\chi_5\rangle + \cdots$$

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Second query (corrupted)

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b/c this is ind. of v_2 to v_k, v_2 to v_k are truly at random.

- Changing oracle #2 barely changes the resulting state after 2 queries. (with high probability)

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Modifications taken from Bubeck et al. can bring n down to $1/\epsilon^4$.

Accelerated Gradient Descent

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Similar proof to the one shown, but the function requires smoothing.

Open Problems

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Yet...

- What is the quantum complexity of convex optimization in small dimensions? $n < \frac{1}{\epsilon^4}$
- What other classes of convex optimization problems get quantum speedups?
- What is the quantum complexity of optimizing the function class

$$f_V(x) = \max\{\langle v_1, x\rangle, \langle v_2, x\rangle, \ldots, \langle v_k, x\rangle\}?$$