Acyclic Vertex Coloring of Graphs of Maximum Degree 6

Satish Varagani Kishore Yadav Kishore Kothapalli V. Ch. Venkaiah
International Institute of Information Technology, Hyderabad, India.
Email: vsatish@students., kyadav@students., kkishore@, venkaiah@iiit.ac.in

Abstract

An acyclic vertex coloring of a graph is a proper vertex coloring such that there are no bichromatic cycles. The acyclic chromatic number of G, denoted $a(G)$, is the minimum number of colors required for acyclic vertex coloring of a graph $G = (V, E)$. For a family $\mathcal{F}$ of graphs, the acyclic chromatic number of $\mathcal{F}$, denoted by $a(F)$, is defined as the maximum $a(G)$ over all the graphs $G \in \mathcal{F}$. In this paper we show that $a(\mathcal{F}) = 12$, where $\mathcal{F}$ is the family of graphs of maximum degree 6 by presenting a linear time algorithm to achieve this bound.

Keywords: Graph theory, analysis of algorithms.

1 Introduction

A proper coloring of the vertices of a graph $G = (V, E)$ is an assignment of colors to the vertices so that no two neighbors get the same color. A proper coloring is said to be acyclic if the coloring does not induce any bichromatic cycles. The acyclic chromatic number of a graph G is denoted $a(G)$, and is the minimum number of colors required to acyclically color the vertices of G.

The concept of acyclic coloring of a graph was introduced by Grunbaum [14] and is further studied in the last two decades in several works, [3, 1, 7, 8, 9, 4] among others. Determination of $a(G)$ is a hard problem even theoretically. For example, Kostochka [16] proves that it is an NP-complete problem to decide for a given arbitrary graph G whether $a(G) \leq 3$.

Given the computational difficulty involved in determining $a(G)$, several authors have looked at acyclically coloring particular families of graphs. In this context, Borodin [7] focuses on the family of planar graphs, the family of planar graphs with ”large” girth [10], 1-planar graphs [9], outer planar graphs [24], d-dimensional grids [13], graphs of maximum degree 3 [14, 23], and of maximum degree 4 [11].

Another direction that has yielded fruits is that of the using the probabilistic method and the Lovasz Local Lemma (LLL) [6]. Using this method, it was shown by Alon et al. [3] that any graph of maximum degree D can be acyclically colored using $O(\Delta^{4/3})$ colors, thus showing that $a(G) \leq \Omega(\Delta^{4/3}/(\log \Delta)^{1/3})$. In the same paper, it was also shown that, as $n \to \infty$, there exist graphs with maximum degree Δ and requiring $\Omega(\Delta^{4/3}/(\log \Delta)^{1/3})$ colors for an acyclic coloring. The above two results are based on the probabilistic method. They further showed that a greedy algorithm exists to acyclically color any graph G with maximum degree Δ using $\Delta^2 + 1$ colors. This was later improved by Albertson et al. [2] to show that $a(G) \leq \Delta(\Delta - 1) + 2$.

Focusing on the family of graphs with a small maximum degree Δ, it was proved by Skulrattanakulchai [23] that $a(G) \leq 4$ for any graph of maximum degree 3. Burnstein [11] showed that $a(G) \leq 5$ for any graph of degree maximum 4. The work of Skulrattanakulchai was extended by Fertin and Raspaud [12] to show that it is possible to acyclically vertex color a graph G of maximum degree Δ using at most $\Delta(\Delta - 1)/2$ colors. In the same paper, it was also shown that for any graph G of maximum degree 5, $a(G) \leq 9$ and there exists a linear time algorithm to acyclically color G using at most 9 colors.Recently, Yadav et al. [29]

extended the work of Skulrattanakulchai [23] to show that any graph of maximum degree 5 can be colored using at most 8 colors.

In this paper, we show that any graph of maximum degree 6 can be acyclically colored using at most 12 colors. Our result thus improves the state-of-the-art for the considered family from 15 colors [12] to 12.Below, we first introduce the notation that is used in the rest of the paper.

1.1 Notation

For a positive integer k, $[k]$ refers to the set of positive integers $\{1, 2, ..., k\}$. For $W \subseteq V(G)$, $N(W)$, is the set of all vertices in $V(G) \setminus W$ that are adjacent to some vertex in W. We borrow notation from [23].

Definition 1.1 *Let $W \in V(G)$. The neighborhood of W, denoted $N(W)$, is the set of all vertices in $V(G) \setminus W$ that are adjacent to some vertex in W. A neighbor of W is a vertex in $N(W)$. $N(v)$ stands for $N(\{v\})$.*

Definition 1.2 *A partial coloring is an assignment of colors to a subset of $V(G)$ such that the colored vertices induce a graph with an acyclic coloring.*

Suppose G has a partial coloring. Let α, β be any two colors. An alternating α, β-path is a path in G with each vertex colored either α or β. An alternating path is an alternating α, β path for some colors α, β. A path is odd or even according to the parity of number of edges it contains. Let v be an uncolored vertex. A color $\alpha \in [12]$ is *available* for v if no neighbor of v is colored α. A color $\alpha \in [12]$ is *feasible* for v if assigning color α to v still results in a partial coloring. Let C_v be a cycle in G containing vertex v. A cycle C_v is α, β dangerous if $C_v - v$ is an even α, β–alternating path. A cycle C_v is dangerous if it is α, β-dangerous for some colors α, β.

Definition 1.3 *We call a vertex v as a* single *vertex if all its colored neighbors receive distinct colors.*

Notice that the above definition also treats a vertex v with some uncolored neighbors but the colored neighbors having distinct colors as a single vertex. The notion of a single vertex is useful because recoloring is easy at single vertices.

1.2 Our Results

In this paper, we show that any graph G with maximum degree bounded by 6 can be acyclically colored using 12 colors. We show this result by extending a partial coloring by one vertex v at a time. During this process, in some scenarios it is required that we recolor some of the vertices already colored so as to make a color feasible for the vertex which we try to color. However, note that this recoloring, if required, is limited to the neighborhood of the neighbors of v, in all cases. Specifically, we show the following lemmata which result in Theorem 1.7.

Lemma 1.4 *Let π be any partial coloring of G using colors in $[12]$ and let v be any uncolored vertex. If v has less than 4 colored neighbors, then there exists a color $\alpha \in [12]$ feasible for v.*

Lemma 1.5 *Let π be any partial coloring of G using colors in $[12]$ and let v be any uncolored vertex. If v has exactly either four or five colored neighbors, then there exists a partial coloring π_1 of G using colors in $[12]$ and a color $\alpha \in [12]$ so that π_1 has the same domain as π, $\pi(x) \neq \pi_1(x)$ implies $x \in N(v)$ or $x \in N(N(v))$ and α is feasible for v under π_1.*

Lemma 1.6 *Let π be any partial coloring of G using colors in $[12]$ and let v be any uncolored vertex. If v has six colored neighbors, then there exists a partial coloring π_1 of G using colors in $[12]$ and a color $\alpha \in [12]$ so that π_1 has the same domain as π, $\pi(x) \neq \pi_1(x)$ implies $x \in N(v)$ or $x \in N(N(v))$, and α is feasible for v under π_1.*

We show that in all the above lemmata, both π_1 and α can be found in $O(1)$ time. Hence, we have:

Theorem 1.7 *The vertices of any graph G of degree at most 6 can be acyclically colored using 12 colors in $O(n)$ time, where n is the number of vertices.*

The rest of the paper is organized as follows. In Section 2, we prove Lemma 1.4. In Section 3 and 4, we prove Lemmata 1.5 and 1.6 respectively. The paper ends with some concluding remarks in Section 5.

2 Proof of Lemma 1.4

We may assume without loss of generality that all 6 neighbors w, x, y, z, t, u exist. Vertex v has at most 6 neighbors.

As maximum number of colored neighbors of v is three assume without loss of generality they are $\{w, x, y\}$. At any time maximum number of available colors for v is nine.Any of $\{w, x, y\}$ can participate at most five $1, \beta$– dangerous cycles and each such C_v cycle contains two neighbors of v. Hence there are at most seven dangerous C_v cycles. Since v has at least nine colors available but at most seven of them are not feasible, at least two colors are feasible for v.

3 Proof of Lemma 1.5

In this section we deal the case where v has exactly either four or five colored neighbors. We deal the proof in several cases depending on the colors of the neighbors of v.

1: v has exactly four colored neighbors: We further subdivide this case into three sub cases.

1.a- When v has at least one uniquely colored neighbor.

When no two neighbors of v have the same color, it can be seen that v has eight available colors, and all of them are feasible for v. Otherwise, assume without loss of generality that the colored neighbors of v are $\{w, x, y, z\}$, and z is uniquely colored. At any time number of available colors for v is at least nine. Any of $\{w, x, y\}$ can participate at most five $1, \beta$– dangerous cycles and each such C_v cycle contain two neighbors of v. Hence, there are at most seven dangerous C_v cycles. Since v has at least nine colors available but at most seven of them are not feasible, at least two colors are feasible for v.

1.b- When there are two colors so that each color appears at exactly two neighbors: Assume without loss of generality that $\pi(w) = \pi(x) = 1$ and $\pi(y) = \pi(z) = 2$. Hence, there are at most five $1, \beta$-dangerous C_v cycles. Similarly, there are at most five $2, \beta$-dangerous C_v cycles. However, only ten colors are available to v. Thus, none of the available colors may be feasible for v. We proceed with the following case distinction.

i- If any of $\{w, x, y, z\}$ is a single vertex: Assume without loss of generality that w has neighbors colored 3, 4, 5, 6 and 7. Define π_1 by setting $\pi_1(w) = 8$ and setting $\pi_1(s) = \pi(s)$ for all other colored vertices s. Then, π_1 is also a partial coloring. Moreover, under π_1, this case converts to sub case [1.a] above.

- ii- If none of $\{w, x, y, z\}$ is a single vertex: In this case, any vertex in $\{w, x, y, z\}$ has at most 4 different colored vertices. Thus, any neighbor of v can participate in at most 4 dangerous C_v cycles, and each C_v cycle contains two neighbors of v. Hence, there are at most 8 dangerous C_v cycles. Since v has 10 available colors but at most 8 of them are not feasible, at least two of them are feasible.

1.c- When there is one color appears at all four colored neighbors: Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = \pi(z) = 1$. Any vertex in $\{w, x, y, z\}$ can participate in at most five dangerous C_v cycles, and each C_v cycle contains two neighbors of v. Hence, there are at most 10 dangerous C_v cycles. Since v has 11 available colors but at most 10 of them are not feasible, at least one color is feasible for v.

2: When v has five colored neighbors, we split the proof into 5 cases.

2.a- When v has at least two uniquely colored neighbors.

If no two neighbors of v have the same color, then v has seven available colors, and all of them are feasible for v. Otherwise assume without loss of generality the colored neighbors are $\{w, x, y, z, u\}$ and $\{z, u\}$ are uniquely colored. At any time at least 8 colors are available for v. Any of $\{w, x, y\}$ can participate in at most five $1, \beta$– dangerous cycles and each such C_v cycle contains two neighbors of v. Hence there are at most seven dangerous C_v cycles. Since v has at least eight colors available but at most seven of them are not feasible, at least one color feasible for v.

2.b- If all the four neighbors have the same color and one is differently colored: Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = \pi(z) = 1$ and $\pi(u) = 2$. Any vertex in $\{w, x, y, z\}$ can participate in at most five dangerous C_v cycles, and each C_v cycle contains two neighbors of v. Hence there are at most 10 dangerous C_v cycles. Thus, none of the available color may be feasible for v. We now deal with two cases. So we make the following case distinction.

- i- If any of $\{w, x, y, z\}$ is a single vertex: Assume without loss of generality that w has neighbors colored 3, 4, 5, 6 and 7. Define π_1 by setting $\pi_1(w) = 8$ and $\pi_1(s) = \pi(s)$ for all other colored vertices s. Then, π_1 is also a partial coloring. Moreover, under π_1, this case converts to Case [2.a] of this section.
- ii- If none of $\{w, x, y, z\}$ is a single vertex: In this case, any vertex in $\{w, x, y, z\}$ has at most 4 differently colored vertices. Thus, any colored neighbor of v can participate in at most 4 dangerous C_v cycles, and each C_v cycle contains two neighbors of v. Hence, there are at most 8 dangerous C_v cycles. Since v has 10 available colors but at most 8 of them are not feasible, at least two of them are feasible.

2.c- If all the five neighbors have the same color: Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = \pi(z) = \pi(u) = 1$. Any vertex in $\{w, x, y, z, u\}$ can participate in at most five dangerous C_v cycles, and each C_v cycle contains two neighbors of v. Hence there are at most 12 dangerous C_v cycles. Thus, none of the available color may be feasible for v. We now deal with two cases.

- i- If any of $\{w, x, y, z, u\}$ is a single vertex: Assume without loss of generality that w has neighbors colored 3, 4, 5, 6, and 7. Define π_1 by setting $\pi_1(w) = 8$ and $\pi_1(s) = \pi(s)$ for all other colored vertices s. Then, π_1 is also a partial coloring. Moreover, under π_1, this case converts to Case 2.b of this section.
- ii- If none of $\{w, x, y, z, u\}$ is a single vertex: In this case, any vertex in $\{w, x, y, z, u\}$ has at most 4 differently colored vertices. Thus, any neighbor of v can participate in at most 4 dangerous C_v cycles, and each C_v cycle contains two neighbors of v. Hence, there are at

most 10 dangerous C_v cycles. Since v has 11 available colors but at most 10 of them are not feasible, at least one of them is feasible.

2.d- When there are three colors so that two of them appear at two vertices each exactly and the other neighbor is colored with the third color:

Assume without loss of generality that $\pi(w) = \pi(x) = 1$, and $\pi(y) = \pi(z) = 2$ and $\pi(u) = 3$. There are at most five $1, \beta$– dangerous C_v cycles for some β involving the path $\{w, v, x\}$ and at most five $2, \beta$–dangerous C_v cycles for some β involving the path $\{y, v, z\}$. Hence, presently, no color is feasible for v. To find a feasible color for v, we make the following case distinction.

i- If any of $\{w, x, y, z\}$ is a single vertex: Assume without loss of generality that w has neighbors colored 4, 5, 6, 7, and 8. Define π_1 by setting $\pi_1(w) = 9$ and setting $\pi_1(s) = \pi(s)$ for all other colored vertices s. Then π_1 is also a partial coloring. Moreover, under π_1, this case converts to Case 2.a of this section.

ii- If none of $\{w, x, y, z\}$ is a single vertex: In this case, any vertex in $\{w, x, y, z\}$ has at most 4 different colored vertices. Thus, any neighbor of v can participate in at most 4 dangerous C_v cycles, and each C_v cycle contains two neighbors of v. Hence, there are at most 8 dangerous C_v cycles. Since v has 9 available colors but at most 8 of them are not feasible, at least one of them is feasible.

2.e- When there are two colors so that one color appears at three neighbors and other color appears at two neighbors: Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = 1$, and $\pi(z) = \pi(u) = 2$.There are at most seven $1, \beta$– dangerous C_v cycles for some β involving the path $\{w, v, x\}$, $\{y, v, x\}$, $\{w, v, y\}$ and at most five $2, \beta$– dangerous C_v cycles for some β involving the path $\{u, v, z\}$. Hence, presently, no color is feasible for v. To find a feasible color for v, we make the following case distinction.

i- If any of $\{w, x, y\}$ is a single vertex: Assume without loss of generality that w has neighbors colored 3, 4, 5, 6, and 7. Define π_1 by setting $\pi_1(w) = 8$ and setting $\pi_1(s) = \pi(s)$ for all other colored vertices s. Then π_1 is also a partial coloring. Moreover, under π_1, this case converts to Case[2.d] of this section.

ii- If any of $\{z, u\}$ is a single vertex: Assume without loss of generality that z has neighbors colored 3, 4, 5, 6, and 7. Define π_1 by setting $\pi_1(w) = 8$ and setting $\pi_1(s) = \pi(s)$ for all other colored vertices s. Then π_1 is also a partial coloring. Moreover, under π_1, this case converts to Case[2.a] of this section.

iii- If none of $\{w, x, y, z, u\}$ is a single vertex: If no available color is feasible for v then there must be six $1, \beta$–dangerous C_v cycles and four $2, \beta$–dangerous C_v cycles. Assume without loss of generality they are 1,3-, 1,4-, 1,5-, 1,6-, 1,7-, 1,8-, and 2,9-, 2,10-, 2,11-, and 2,12-dangerous cycles. This implies that the neighbors of z are colored with colors 9, 10, 11, 12 and another neighbor should be colored any of $\{9, 10, 11, 12\}$. Let that color be 9 and let the like neighbors of z be z_1, z_2, i.e., z_1 and z_2 are colored with color 9. The available colors for recoloring z is any of $\{3, 4, 5, 6, 7, 8\}$. Suppose we define π_1 by setting $\pi_1(w) = k$ where k is the color missing from $\{3, 4, 5, 6, 7, 8\}$ in the colored neighbors of z_1 and z_2 and setting, $\pi_1(s) = \pi(s)$ for all other colored vertices s. Moreover, under π_1 this case converts to Case 2.a of this section.

4 Proof of Lemma 1.6

We divide proof of Lemma 1.6 into seven sub-cases.

1: If at most three neighbors of v have the same color and all other neighbors are differently colored. We show that in this case, there is always a feasible color for v without recoloring any colored vertex. To this end, we break the proof into three scenarios.

1.a- No two neighbors of v have the same color: In this case, v has six available colors, and all of them are feasible for v.

1.b- If exactly two neighbors of v colored with the same color and the remaining neighbors are colored differently: Assume without loss of generality that $\pi(w) = \pi(x) = 1$, $\pi(y) = 2$, $\pi(z) = 3$, $\pi(t) = 4$, and $\pi(u) = 5$. Thus colors [6-12] are available for v. Any dangerous cycle involving v is $1, \beta$–dangerous for some β and it involves the path $\{w, v, x\}$. As $\Delta = 6$, no more than 5 colors can be the color β in any $1, \beta$-dangerous cycle involving v. Hence there are at least 2 colors are feasible for v.

1.c- When there are four colors so that one color appears at three neighbors and other three colors appears at exactly one neighbor: Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = 1$, $\pi(z) = 2$, $\pi(t) = 3$, and $\pi(u) = 4$. Thus colors [5-12] are available for v. Any of $\{w, x, y\}$ can participate at most five $1, \beta$– dangerous cycles and each such C_v cycle contains two neighbors of v. Hence there are at most seven dangerous C_v cycles. Since v has eight colors available, at least one of the color is available for v.

2: If two neighbors of v have the same color and another two are colored with the same, but different from above, color, and the other two are differently colored: Assume without loss of generality that $\pi(w) = \pi(x) = 1$, $\pi(y) = \pi(z) = 2$, $\pi(u) = 3$, and $\pi(t) = 4$. There are at most five $1, \beta$– dangerous C_v cycles for some β involving the path $\{w, v, x\}$ and at most five $2, \beta$– dangerous C_v cycles for some β involving the path $\{y, v, z\}$. Hence, presently, no color is feasible for v. To find a feasible color for v, we proceed by making following case distinction.

2.a- If any of $\{w, x, y, z\}$ is a single vertex: Assume without loss of generality that w has neighbors colored $5, 6, 7, 8, 9$. Define π_1 by setting $\pi_1(w) = 10$ and setting $\pi_1(s) = \pi(s)$ for all other colored vertices s. Then π_1 is also a partial coloring. Moreover, under π_1, this case converts to Case 1 of this section.

2.b- If none of $\{w, x, y, z\}$ is a single vertex: For no available color to be feasible for v there must be four $1, \beta$-dangerous C_v cycles and four $2, \beta$-dangerous C_v cycles. Assume without loss of generality that there are $1, 5$, $1, 6$, $1, 7$, $1, 8$, and $2, 9$-, $2, 10$-, $2, 11$-, and $2, 12$-dangerous C_v cycles. This implies that neighbors of w and x are colored with colors in $\{5, 6, 7, 8\}$ and another neighbor should be colored with any of $\{5, 6, 7, 8\}$. Let that color be 5 and let the like neighbors of w be w_1, w_2, i.e., w_1 and w_2 are colored with color 5. If none of w_1 or w_2 have a neighbor colored with color 1, apart from w, then the color 5 is feasible for v since there is no possibility of $1, 5$–dangerous C_v cycles. Otherwise, without loss of generality, let w_1 have a neighbor colored with color 1. We distinguish between two cases. If the neighborhood of w_1 has vertices colored with colors among $\{1, 9, 10, 11, 12\}$, we define π_1 as $\pi_1(w)$ to be any of $\{9, 10, 11, 12\}$, $\pi_1(w_1)$ to be any of 3 or 4, and π_1 agrees with π at all other colored vertices. Under this definition of π_1, which is a partial coloring, this case converts to case 1 above. On the other hand, if some color $k \in \{9, 10, 11, 12\}$ is missing in the neighborhood of w_1, we define π_1 by setting $\pi_1(w) = k$ and $\pi_1(s) = \pi(s)$ for all other colored vertices s. Then, π_1 is also a partial coloring, and under π_1 this case converts to Case 1 above.

3: If three neighbors have the same color and another two are colored with the same, but different from above, color and the remaining neighbor is differently colored: Assume without loss of generality that

$\pi(w) = \pi(x) = \pi(y) = 1$, $\pi(z) = \pi(t) = 2$, and $\pi(u) = 3$. Any of $\{w, x, y\}$ can participate in at most five $1, \beta$-dangerous cycles and each such C_v cycle contains two neighbors of v. Hence, there are at most seven $1, \beta$-dangerous C_v cycles and at most five $2, \beta$-dangerous C_v cycles. Hence, none of the available colors may be feasible for v. So we proceed by making the following case distinction.

3.a- If any of $\{w, x, y\}$ is a single vertex: Assume without loss of generality that w has neighbors colored 4, 5, 6, 7, and 8. Define π_1 by setting $\pi_1(w) = 9$ and setting $\pi_1(s) = \pi(s)$ for all other colored vertices s. Then π_1 is also a partial coloring. Moreover, under π_1 this case converts to Case 2 of this section.

3.b- If any of $\{z, t\}$ is a single vertex: Assume without loss of generality that z has neighbors colored 4, 5, 6, 7, and 8. Define π_1 by setting $\pi_1(z) = 9$ and setting $\pi_1(s) = \pi(s)$ for all other colored vertices s. Then π_1 is also a partial coloring. Moreover, under π_1 this case converts to subcase 1.c of this section.

3.c- If none of $\{w, x, y, z, t\}$ is a single vertex: We divide into two sub case.

i. If any of $\{w, x, y, z, t\}$ has four differently colored neighbors: Then any of $\{w, x, y, z, t\}$ can participate in at most four $1, \beta$-dangerous cycles and each such C_v cycle contains two neighbors of v. Hence, there are at most six $1, \beta$-dangerous C_v cycles. Similarly, there are at most four $2, \beta$-dangerous C_v cycles. Hence, none of the available colors may be feasible for v. Assume without loss of generality that w has neighbors colored with colors among 4, 5, 6, 7. The other neighbor of w should be colored with one among $\{4, 5, 6, 7\}$. Let it be 7 and the alike neighbors of w be w_1 and w_2. When any one of the colors in $\{8, 9, 10, 11, 12\}$ is found missing in the colored neighborhood of w_1 or w_2 then define π_1 by setting $\pi_1(w) = k$, where k is a missing color and $\pi_1(s) = \pi(s)$ for all other colored vertices s. When no color from $\{8, 9, 10, 11, 12\}$ is found missing among the colored neighborhood of w_1 and w_2, then define π_1 by setting $\pi_1(w_1) = 2$, $\pi_1(w) = 9$, and π_1 agrees with π at all other colored vertices. Under this definition of π_1, which is a partial coloring, this case converts to Case 2 above.

ii. On the other hand, when all vertices in $\{w, x, y, z, u\}$ have less than four differently colored neighbors, then any of $\{w, x, y\}$ can participate in at most three $1, \beta$–dangerous cycles and each such C_v cycle contains two neighbors of v. Hence, there are at most four $1, \beta$-dangerous C_v cycles. Similarly, there are at most three $2, \beta$-dangerous C_v cycles. Hence there are at least two available colors that are feasible for v.

4: When there are three colors with each color appearing at exactly two neighbors: Assume without loss of generality that $\pi(w) = \pi(x) = 1$, $\pi(y) = \pi(z) = 2$, and $\pi(t) = \pi(u) = 3$. There are at most five $1, \beta$–dangerous C_v cycles for some β involving the path $\{w, v, x\}$, at most five $2, \beta$–dangerous C_v cycles for some β involving the path $\{y, v, z\}$ and at most five $3, \beta$–dangerous C_v cycles for some β involving the path $\{t, v, u\}$. Since v has 9 available colors and all of them can be involved in dangerous C_v cycles, no color may be feasible for v. We proceed in the following way by making a case distinction.

4.a- If any of the vertices in $\{w, x, y, z, t, u\}$ are single: Assume without loss of generality that w has neighbors colored with colors 4, 5, 6, 7, and 8. Define π_1 by setting $\pi_1(w) = 9$ and setting $\pi_1(s) = \pi(s)$ for all other colored vertices s. Then π_1 is also a partial coloring. Moreover, under π_1 this case converts to Case 2 (subcase) of this Section.

4.b- If none of $\{w, x, y, z, u\}$ is a single vertex: We make the following case distinction based on the colors in the neighborhood of the vertices in $\{w, x, y, z, t, u\}$.

i. If all of $\{w, x, y, z, t, u\}$ have less than three differently colored neighbors: In this case, it can be shown that there is at least one color that is feasible for v.

ii. If all of $\{w, x, y, z, t, u\}$ have only three differently colored neighbors: Let us consider w and the like colored neighbors of w be w_1, w_2, and w_3. Without loss of generality, let w have neighbors colored with colors in $\{4, 5, 6\}$. There are at most seven dangerous cycles can be formed using $\{w_1, w_2, w_3\}$. Hence one color is available for recoloring w among colors in $\{7, 8, 9, 10, 11, 12, \ 2, 3\}$. Then define a partial coloring π_1 by setting $\pi_1(w)$ as any of the available color and $\pi_1(s) = \pi(s)$ for all other colored vertices s. Then $\pi_1$is also a partial coloring. Moreover, under π_1 this case converts to Case 3 above when $\pi_1(w) = 2$ or 3, or else it converts into Case 2 above.

iii. If any of $\{w, x, y, z, t, u\}$ has four differently colored neighbors. Let it be w. Assume without loss of generality that w has neighbors colored with colors in $\{4, 5, 6, 7\}$ and the other neighbor is colored with a color among the colors $\{4, 5, 6, 7\}$. Let it be 7 and the alike neighbors of w be w_1 and w_2. If any of $\{8, 9, 10, 11, 12\}$ is found missing in the neighborhood of w_1 or w_2 then define π_1 by setting $\pi_1(w) = k$ where k is the color missing and $\pi_1(s) = \pi(s)$ for all other colored vertices s. Then π_1 is also a partial coloring. Moreover, under π_1 this case converts to Case 2 of Section 4. If none of $\{8, 9, 10, 11, 12\}$ is found missing in the neighborhood of w_1 and w_2, then define π_1 by setting $\pi_1(w_1) = 2$, $\pi_1(w) = 9$, and π_1 agrees with π at all other colored neighbors. Notice that π_1 is also a partial coloring and this case converts to case 2 above.

5: When there are two colors with each color appearing at three neighbors: Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = 1$ and $\pi(z) = \pi(t) = \pi(u) = 2$. Any of $\{w, x, y\}$ can participate in at most five $1, \beta$-dangerous cycles and each such C_v cycle contains two neighbors of v. Hence, there are at most seven $1, \beta$-dangerous C_v cycles. Similarly, there are at most seven $2, \beta$-dangerous C_v cycles. Hence, none of the available colors may be feasible for v. We proceed in the following way by making a case distinction.

5.a- If any of neighbors of v is a single vertex: Assume without loss of generality that w has neighbors colored with colors 4, 5, 6, 7, and 8. Define π_1 by setting $\pi_1(w) = 9$ and setting $\pi_1(s) = \pi(s)$ for all other colored vertices s. Then π_1 is also a partial coloring. Moreover, under π_1 this case converts to Case 3 of this Section.

5.b- -If none of the neighbors of v is a single vertex: We distinguish between the following cases.

i. When all vertices in $\{w, x, y, z, t, u\}$ have less than four differently colored neighbors there it can be shown that at least two feasible colors for coloring of v, thereby completing the case.

ii. When any of $\{w, x, y, z, t, u\}$ has four differently colored neighbors: Let it be w and assume without loss of generality that w has neighbors colored with colors 3, 4, 5, and 6. The other neighbor of w should be colored among colors in $\{3, 4, 5, 6\}$. Let it be 6 and the alike neighbors of w be w_1 and w_2. There are at most five dangerous cycles (say 6,7-, 6,8-, 6,9-, 6,10-, and 6,11-) that can be formed. Hence, we recolor w with the remaining color, 12 in this case. So, define π_1 by setting $\pi_1(w) = 12$ and $\pi_1(s) = \pi(s)$ for all other colored vertices s. Then π_1 is also a partial coloring and under π_1 this case converts to Case 3 above.

6 : If at least four neighbors are colored with same color and the remaining colored neighbors are colored with distinct (different from the above) colors: We treat this case by further looking at the following three subcases.

6.a- When there are three colors so that one color appears at four neighbors and remaining two colors appear exactly at one neighbor each: Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = \pi(z) = 1$, $\pi(u) = 2$ and $\pi(t) = 3$. There can be at most ten $1, \alpha$-dangerous cycles and no color is feasible for v.

If none of $\{w, x, y, z\}$ is a single vertex, it can be shown that there exists at least one color that is feasible for v. On the other hand, if all the vertices in $\{w, x, y, z\}$ are single, assume without loss of generality that w is a single vertex. Assume without loss of generality that w has neighbors colored with colors 4, 5, 6, 7, and 8. Define π_1 by setting $\pi_1(w) = 9$ and setting $\pi_1(s) = \pi(s)$ for all other colored vertices t. Then π_1 is also a partial coloring. Moreover, under π_1 this case converts to subcase 1.c of this Section.

6.b- When there are two colors so that one color appears at five neighbors and remaining color is appears at one neighbor: Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = \pi(z) = \pi(u) = 1$ and $\pi(t) = 2$.We proceed in the following way by making a case distinction.

If any of $\{w, x, y, z, u\}$ is a single vertex, assume without loss of generality that w is a single vertex. Let w have neighbors colored with colors 3, 4, 5, 6, and 7. Define π_1 by setting $\pi_1(w) = 8$ and setting $\pi_1(s) = \pi(s)$ for all other colored vertices s. Then π_1 is also a partial coloring. Moreover, under π_1 this case converts to Case (6.a) above.

On the other hand, if none of $\{w, x, y, z, u\}$ is a single vertex, we proceed as follows. For no available color feasible for v all of $\{w, x, y, z, u\}$ must contain four differently colored neighbors. Assume without loss of generality that w has neighbors colored with colors in $\{3, 4, 5, 6\}$ and the other neighbor is colored with a color among the colors $\{3, 4, 5, 6\}$. Let it be color 3 and the alike neighbors of w be w_1 and w_2. We recolor w as follows. We define π_1 by setting $\pi_1(w) = k$ where k is the color missing from $\{7, 8, 9, 10, 11, 12\}$ in the neighbors list of w_1 and w_2 and setting $\pi_1(s) = \pi(s)$ for all other colored vertices s. Thus, π_1 is a partial coloring and, under π_1 this case converts to Case (6.a) above.

6.c- When all six neighbors of v have the same color: Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = \pi(z) = \pi(u) = \pi(t) = 1$. We proceed in the following way by making a case distinction.

If any of $\{w, x, y, z, u\}$ is a single vertex, assume without loss of generality that w has neighbors colored with colors 2, 3, 4, 5, and 6 . We recolor w by definining π_1 as $\pi_1(w) = 7$ and $\pi_1(s) = \pi(s)$ for all other colored vertices t. Then π_1 is also a partial coloring. Moreover under π_1 this case converts to above Case (6. b).

On the other hand, if none of $\{w, x, y, z, u\}$ is a single vertex, it can be shown that for no available color feasible for v at least one of $\{w, x, y, z, u\}$ must contain four different neighbors. Assume without loss of generality that w has neighbors colored with colors in $\{2, 3, 4, 5\}$ and the other neighbor of w is colored with a color among the colors $\{2, 3, 4, 5\}$. Let it be color 3 and the alike neighbors of w be w_1 and w_2. We now recolor w as follows. We define π_1 as $\pi_1(w) = k$ where k is the color missing from $\{6, 7, 8, 9, 10, 11, 12\}$ in the neighbors list of w_1 and $w_2$and setting, $\pi_1(s) = \pi(s)$ for all other colored vertices s. Thus, π_1 is a partial coloring and, under π_1 this case converts to Case (6.b) above.

7 : When there are two colors so that one color appears at four neighbors and the other color appears at two neighbors: Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = \pi(z) = 1$, and $\pi(u) = \pi(t) = 2$. There can be at most ten $1, \alpha$-dangerous cycles and at most five $2, \beta$-dangerous cycles. Thus, all the ten available colors may not be feasible for v. So, we proceed in the following way by making a case distinction.

7.a- Any of $\{w, x, y, z\}$ is a single vertex: Assume without loss of generality that w has neighbors colored with colors 3, 4, 5, 6 and, 7. Define π_1 by setting $\pi_1(w) = 8$ and setting $\pi_1(s) = \pi(s)$ for all other colored vertices s. Then π_1 is also a partial coloring. Moreover, under π_1 this case converts to Case 3 of this section.

7.b- Any of $\{t, u\}$ is a single vertex: Assume without loss of generality that t has neighbors colored with colors 3, 4, 5, 6 and, 7. We recolor the vertex t by defining π_1 as $\pi_1(t) := 8$ and $\pi_1(s) := \pi(s)$ for all other colored vertices s. Then π_1 is also a partial coloring. Moreover, under π_1, this case converts to Case (6.a) above.

7.c- If none of $\{w, x, y, z, t, u\}$ is a single vertex: We proceed in the following way by making a case distinction on the number of $2, \beta$-dangerous cycles.

i. If there are four $2, \beta$-dangerous C_v cycles: Assume without loss of generality they are 2,9-, 2,10-,2,11-, and 2,12- dangerous C_v cycles. This implies that four neighbors of u and t are colored with colors 9, 10, 11, and 12 and another neighbor should be colored with any color among $\{9, 10, 11, 12\}$. Let it be 9 and the like colored neighbors of u be u_1, u_2, i.e., u_1 and u_2 are colored with color 9. When any of $\{3, 4, 5, 6, 7, 8\}$ is missing in the neighborhood of u_1 or u_2, we define π_1 by setting $\pi_1(u) = k$ where k is one among the missing colors, and setting $\pi_1(s) = \pi(s)$ for all other colored vertices s. Then, π_1 is also a partial coloring. Moreover, under π_1 this case converts to Case 6.a above.

ii. If there are at most three $2, \beta$-dangerous C_v cycles: For no available color to be feasible for v there must be at least seven $1, \beta$- dangerous C_v cycles. Assume without loss of generality there are 1,3-, 1,4-, 1,5-, 1,6-, 1,7-, 1,8- and 1,9-dangerous C_v cycles. This implies that at least one of $\{w, x, y, z\}$ must contain four differently colored neighbors. Let it be w. Assume that four neighbors of w are colored with colors 3, 4, 5, and, 6 and another neighbor of w is colored with any of $\{3, 4, 5, 6\}$. Let it be 3 and the like colored neighbors of w be w_1, w_2, i.e., w_1 and w_2 are colored with color 3. When any of the colors in $\{7, 8, 9, 10, 11, 12\}$ is missing in the neighborhood of any of w_1 or w_2, we define π_1 by setting $\pi_1(w) = k$, where k is one among the missing colors, and setting $\pi_1(s) = \pi(s)$ for all other colored vertices s. Then, π_1 is also a partial coloring. Moreover, under π_1 this case converts to Case 3 of this section .

5 Conclusions

In this paper, we have presented a polynomial time algorithm to acyclically color the vertices of graphs whose maximum degree is bounded by 6. The algorithm improves the state-of-the-art by 3 colors by a careful consideration of the various cases.

References

[1] M. O. Albertson and D. M. Berman. The acyclic chromatic number. Proc. *Seventh Southeastern Conference on Combinatorics, Graph Theory and Computing*, Utilitas Mathematica Inc., Winniepeg, Canada 51–60, 1976.

[2] M. O. Albertson, G. G. Chappell, H. A. Kierstead, A. Kundgen, and R. Ramamurthi. Coloring with no 2-colored P_4's, *The Electronic Journal of Combinatorics*, 11(1), 2004.

[3] N. Alon, C. McDiarmid, and B. Reed. Acyclic colouring of graphs, *Random Str. and Alg.*, 2:277–288, 1991.

[4] N. Alon, B. Sudakov, and A. Zaks. Acyclic Edge Colorings of Graphs. *J. Graph Th.*, 37:157–167, 2001.

[5] N. Alon and A. Zaks. Algorithmic aspects of acyclic edge colorings. *Algorithmica*, 32:611–614, 2002.

[6] J. Beck. An algorithmic approach to the Lovasz Local lemma. *Random Structures and Algorithms*, 2:343–365, 1991.

[7] O. V. Borodin. On Acyclic colorings of planar graphs. *Discrete Math.*, 25:211–236, 1979.

[8] O. V. Borodin, D. G. Fon-Der Flaass, A. V. Kostochka, A. Raspaud, and E. Sopena. Acyclic list 7-coloring of planar graphs. *J. Graph Theory*, 40(2):83–90, 2002.

[9] O. V. Borodin, A. V. Kostochka, A. Raspaud, and E. Sopena. Acyclic coloring of 1-planar graphs. *Diskretnyi Analiz i Issledivanie Operacii*, Series 1 6(4):20–35, 1999.

[10] O. V. Borodin, A. V. Kostochka, and D. R. Woodwall. Acyclic coloring of planar graphs with large girth. *J. London Math. Soc.*, 60(2):344–352, 1999.

[11] M. I. Burnstein. Every 4-valent graph has an acyclic 5-coloring. *Soobsc Akad. Nauk Grucin*, 93:21–24 (in Russian), 1979.

[12] G. Fertin and A. Raspaud. Acyclic colorings of graphs of maximum degree Δ. in *European Conference on Combinatorics Graph Theory, and Applications*, pp. 389-396. 2005.

[13] G. Fertin, E. Godard, and A. Raspaud. Acyclic and k-distance coloring of the grid. *Information Processing Letters*, 87(1):5158, 2003.

[14] B. Grunbaum. Acyclic Colorings of planar graphs. *Israel J Math*, 14:390–408, 1973.

[15] A. V. Kostochka, E. Sopena, and X. Zhu. Acyclic and oriented chromatic numbers of graphs. *J. Graph Theory*, 24(4):331–340, 1997.

[16] A. V. Kostochka Upper bounds on the chromatic functions of graphs. Ph. D. Thesis, Novosibirsk, Russia, 1978.

[17] A. Kotzig. Hamilton graphs and Hamilton circuits. Theory of Graphs and Its Applications (Proc. Sympos. Smolenice 1963), *Nakl. CSAV, Praha* 62–82, 1964.

[18] H. L. Montgomery. *Topics in Multiplicative Number Theory*, Lecture Notes in Mathematics, 227, Springer, Berlin, 1971.

[19] M. Molloy and B. Reed. Further algorithmic aspects of Lovasz local lemma. In *30th Annual ACM Symposium on Thery of Computing*, 524–529, 1998.

[20] N. Muthu, N. Narayanan, C. R. Subramanian. Improved bounds on Acyclic Edge Coloring. In *Elec. N. in Disc. Math.*, 19:171–177, 2005.

[21] J. Nesetril and N. C. Wormald. The acyclic edge chromatic number of a random d-regular graph is d+1. J. Graph Th., 49(1):69–74, 2005

[22] A. Prasant Gopal, Kishore Kothapalli, V. Ch. Venkaiah, C. R. Subramanian, W. D. Wallis. Lexicographically smallest and other one-factorizations of complete graphs. Communicated.

[23] S. Skulrattanakulchai. Acyclic Colorings of Subcubic Graphs. *Inf. Proc. Lett.*, 92:161–167, 2004.

[24] E. Sopena. The chromatic number of oriented graphs. Mathematical Notes, **25**:191205, 1997.

[25] C. R. Subramanian. Analysis of a heuristic for acyclic edge colouring. *Information Processing Letters*, 99:227–229, 2006.

[26] D. G. Wagner. On the perfect one - factorization conjecture. *Discrete Mathematics*, 104:211–215, 1992.

[27] W. D. Wallis. *One Factorizations*. Kluwer Academic Publisher, 1997.

[28] D. West. *Introduction to Graph Theory*. Prentice-Hall, 2001.

[29] Kishore Yadav and Satish Varagani, Kishore Kothapalli, and V. Ch. Venkaiah. Acyclic Vertex Coloring of Graphs of Maximum Degree 5, *Proc. of the International Conference on Graph Theory and its Applications*, 2008, Coimbatore, India. (Under submission to Discrete Mathematics, 2009).