Acyclic Vertex Coloring of Graphs of Maximum Degree Δ

Satish Varagani Kishore Kothapalli

International Institute of Information Technology

Gachibowli, Hyderabad, India – 500 032.

Email:{vsatish@research., kkishore@}iiit.ac.in

V. Ch. Venkaiah

C.R.Rao's Advanced Institute of Mathematics, Statistics and Computer Science

Hyderabad, India – 500 032.

Email:venkaiah@hotmail.com

Abstract

An acyclic vertex coloring of a graph is a proper vertex coloring such that there are no bichromatic cycles. The acyclic chromatic number of G, denoted $a(G)$, is the minimum number of colors required for acyclic vertex coloring of graph $G = (V, E)$. In this paper we show that for any graph G with maximum degree Δ, $a(G) \leq \frac{3\Delta^2+4\Delta+8}{8}$. This improves the known result of Fertin and Raspaud [11] by a factor of 4/3, while using similar techniques. Our proof investigates the colors in 3-neighborhood, as opposed to the 2-neighborhood in the case of [11].

SECTION: A – Combinatorics, Graph Theory and Discrete Mathematics.

1 Introduction

A proper coloring of the vertices of a graph $G = (V, E)$ is an assignment of colors to the vertices so that no two neighbors get the same color. A proper coloring is said to be *acyclic* if the coloring does not induce any bi-chromatic cycles. The acyclic chromatic number of a graph G is denoted $a(G)$, and is the minimum number of colors required to acyclically color the vertices of G.

The concept of acyclic coloring of a graph was introduced by Grünbaum [13] and is further studied in the last two decades in several works, [3, 1, 6, 7, 8, 4] among others. Determination of $a(G)$ is a hard problem even theoretically. For example, Kostochka [14] proves that it is an NP-complete problem to decide for a given arbitrary graph G whether $a(G) \leq 3$.

Given the computational difficulty involved in determining $a(G)$, several authors have looked at acyclically coloring particular families of graphs. In this context, Borodin [6] focuses on the family of planar graphs, the family of planar graphs with "large" girth [9], 1-planar graphs [8], outer planar graphs [16], d-dimensional grids [12], graphs of maximum degree 3 [13, 15], and of maximum degree 4[10].

Another direction that has yielded fruits is that of using the probabilistic method and the Lovász Local Lemma (LLL) [5]. Using this method, it was shown by Alon et al. [3] that any graph of maximum degree Δ can be acyclically colored using $O(\Delta^{4/3})$ colors, thus showing that $a(G) \leq O(\Delta^{4/3})$. In the same paper, it was also shown that, as $n \to \infty$, there exist graphs with maximum degree Δ and requiring $\Omega(\Delta^{4/3}/(\log \Delta)^{1/3})$ colors for an acyclic coloring. The above two results are based on the probabilistic method. They further showed that a greedy algorithm exists to acyclically color any graph G with maximum degree Δ using $\Delta^2 + 1$ colors. This was later improved by Albertson et al. [2] to show that $a(G) \leq \Delta(\Delta - 1) + 2$.

Focusing on the family of graphs with a small maximum degree Δ, it was proved by Skulrattanakulchai [15] that $a(G) \leq 4$ for any graph of maximum degree 3. Burnstein [10] showed that $a(G) \leq 5$ for any graph of degree maximum 4. The work of Skulrattanakulchai was extended by Fertin and Raspaud [11] to show that it is possible to acyclically vertex color a graph G of maximum degree Δ using at most $\Delta(\Delta - 1)/2$ colors. This improves the known result of [2] by a factor of 2. In the same paper, it was also shown that for any graph G of maximum degree 5, $a(G) \leq 9$ and there exists a linear time algorithm to acyclically color G using at most 9 colors. For graphs with $\Delta \leq 5$, it was recently shown by [18] that 8 colors suffice. Similarly, for $\Delta \leq 6$, it is shown that 12 colors suffice to arrive at an acyclic vertex coloring [19].

In this paper, we improve the result of [11] to show that any graph of maximum degree Δ can be acyclically colored using $C(\Delta) = \frac{3\Delta^2+4\Delta+8}{8}$ colors. For $\Delta \geq 8$, the number of colors used by our approach is smaller than the bound obtained by Fertin and Raspaud [11]. Below, we first introduce the notation that is used in the rest of the paper.

1.1 Notation

For a positive integer k, $[k]$ refers to the set of positive integers $\{1, 2, ..., k\}$. We stick to standard graph theoretic notation (cf. [17]) for terms not defined here. We use notation from [15, 11], and repeat it for sake of clarity. We start with the following definition.

Definition 1.1 *Let $W \subseteq V(G)$. The neighborhood of W, denoted $N(W)$, is the set of all vertices in $V(G) \setminus W$ that are adjacent to some vertex in W. A neighbor of W is a vertex in $N(W)$, $N(v)$ stands for $N(\{v\})$.*

Definition 1.2 *A partial coloring is an assignment of colors to a subset of $V(G)$ such that the colored vertices induce a graph with an acyclic and proper coloring.*

Suppose G has a partial coloring. Let α, β be any two colors. An alternating α, β-path is a path in G with each vertex colored either α or β. An alternating path is an alternating α, β-path for some colors α, β. A path is odd or even according to the parity of number of edges it contains. Let v be an uncolored vertex. A color $\alpha \in [C(\Delta)]$ is *available* for v if no neighbor of v is colored α. A color $\alpha \in [C(\Delta)]$ is *feasible* for v if assigning color α to v still results in a partial coloring. (Thus feasibility implies availability but not the other way around). Let C_v be a cycle in G containing vertex v. A cycle C_v is α, β*-dangerous* if $C_v - v$ is an even α, β-alternating path. A cycle C_v is dangerous if it is α, β-dangerous for some colors α, β. If there are more than one α, β-dangerous cycles through v for fixed α, β we consider them as the same type of dangerous cycles.

We now introduce the following definition that allows us to study the colors in the neighborhood of a vertex v in G.

Definition 1.3 *Any colored neighbor of v whose color appears exactly once in the neighborhood of v is called a* single vertex*. Similarly, any colored neighbor of v whose color appears at at least two neighbors of v is called a* non single vertex*.*

The notion of single vertices is important as they cannot participate in dangerous cycles.

1.2 Our Results

In this paper, we show that any graph G with maximum degree Δ can be acyclically colored using $C(\Delta)$ colors. We show this result by extending a partial coloring by one vertex at a time. During this process, in some scenarios it is required that we recolor some of the vertices already colored so as to make a color feasible for the vertex which we try to color. However, note that this recoloring, if required, is limited to the neighborhood of v, in all cases. Moreover, notice that, without loss of generality, we can always consider that v has Δ neighbors and that all these Δ neighbors are colored. We show the following lemmas which result in Theorem 1.6.

Lemma 1.4 *Let π be any partial coloring and v be any uncolored vertex. Let v has neighbors $v_1, v_2, \cdots, v_\Delta$ that are all colored. If all the non-single neighbors of v have at most $3\Delta/4$ colors that appear in their neighborhood, then a color α that is feasible for v exists. Moreover, such a color α can be found in polynomial time.*

Lemma 1.5 *Let π be any partial coloring and v be any uncolored vertex. Let v has neighbors $v_1, v_2, \cdots, v_\Delta$ that are all colored. If any of the non-single vertices of v has at least $3\Delta/4+1$ colors that appear in its neighborhood, then a partial coloring π' and a color α can be found so that:*

- *The domain of π' is the same as that of π,*
- *$\pi(x) \neq \pi'(x)$ whenever $x \in N(v)$, and x is a non-single vertex with at least $3\Delta/4+1$ colors appearing in its neighborhood, and*
- *α is feasible for v under π'.*

Putting together the above two lemmata, we get the following theorem, our main result.

Theorem 1.6 (Main Theorem) *The vertices of any graph G of degree at most Δ can be acyclically colored using $C(\Delta)$ colors in $O(n\Delta^3)$ time, where n is the number of vertices.*

The above result is possible because when we try to find a feasible color for an yet uncolored vertex v, we investigate the 3-neighborhood of v. This allows us to place a better bound on the number of dangerous cycles involving v so that a feasible color for v can be found within $C(\Delta)$ colors.

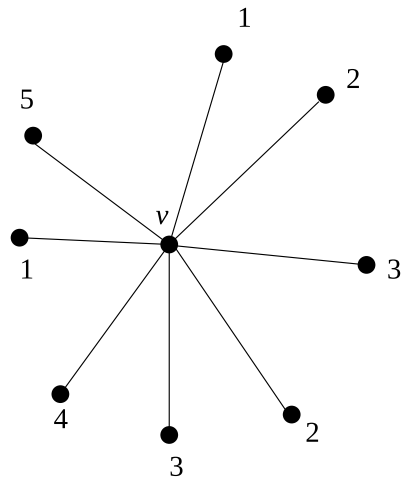

Figure 1: A weaker proof that bounds the number of dangerous cycles considering the colors in the neighborhood of v. In the above picture, the numbers indicate the colors. We consider $\Delta = 8$ and two neighbors of v are single vertices.

In Section 2, we prove Lemmata 1.4, 1.5. The paper ends with some concluding remarks in Section 3.

2 Proof of the Main Theorem

To demonstrate our technique we first show a weaker result using $C^{'}(\Delta) = \frac{\Delta^2}{2} + 1$ colors. Notice that this result is actually slightly weaker than the result obtained in [11] by about $\frac{\Delta}{2}$ colors. However, this proof serves to illustrate our technique.

Theorem 2.1 *Any graph G of maximum degree Δ can be colored acyclically using at most $C^{'}(\Delta) = \frac{\Delta^2}{2} + 1$ colors in polynomial time.*

Proof. Our algorithm colors one uncolored vertex in every iteration. Initially all vertices are uncolored. Let v be the vertex that is being colored in an iteration. Without loss of generality assume that all the neighbors of v are colored, let $v_1, v_2, v_3, \cdots, v_{\Delta-1}, v_\Delta$ be the neighbors of v.

We consider the colors of neighbors of v to find a feasible color for v as follows. Let $c_1, c_2, \cdots, c_\ell$ be the colors that appear more than once in neighborhood of v and, $n_1, n_2, n_3, \cdots, n_\ell$ refer to the number of vertices colored $c_1, c_2, \cdots, c_\ell$ respectively. It can be inferred that $\Delta - \sum_{i=1}^{\ell} n_i$ colors appear in the neighbors of v at exactly one vertex each. See Figure 1 for an example.

The importance of $c_1, c_2, \cdots, c_\ell$ arises as these may be involved in dangerous cycles through v. To find a feasible color for v we place an upper bound on the number of such dangerous cycles as follows.

Consider two vertices v_i, v_j, where $1 \leq i, j \leq \Delta$, such that v_i and v_j are colored with the same color. Then, the number of possible $v_i{-}v{-}v_j$ dangerous cycles is at most $\Delta - 1$. Extending over all such like colored neighbors of v, we get that there at most $\sum_{i=1}^{\ell} \lfloor \frac{n_i(\Delta-1)}{2} \rfloor$ possible dangerous cycles through v. Hence the number of colors that are infeasible for for v is

$$\underbrace{\sum_{i=1}^{\ell} \lfloor \frac{n_i(\Delta-1)}{2} \rfloor}_{\text{number of possible dangerous cycles through } v.} \quad + \quad \underbrace{\Delta - \sum_{i=1}^{\ell} n_i + \ell}_{\text{number of colors in neighborhood of } v.}$$

The above can be simplified as follows.

$$\begin{aligned}
\sum_{i=1}^{\ell} \lfloor \frac{n_i(\Delta-1)}{2} \rfloor + \Delta - \sum_{i=1}^{\ell} n_i + \ell &\leq \sum_{i=1}^{\ell} \frac{n_i(\Delta-1)}{2} + \Delta - \sum_{i=1}^{\ell} n_i + \ell \\
&\leq (\frac{\Delta-1}{2} - 1) \sum_{i=1}^{\ell} n_i + \Delta + \ell \\
&\leq \Delta(\frac{\Delta-1}{2} - 1) + \Delta + \ell, \text{ as } \textstyle\sum_{i=1}^{\ell} n_i \text{ is at most } \Delta, \text{ when} \\
&\quad\ \text{all neighbors participate in dangerous cycles.} \\
&= \frac{\Delta(\Delta-1)}{2} + \ell \\
&\leq \frac{\Delta(\Delta-1)}{2} + \frac{\Delta}{2}, \text{ as when } n_i \geq 2 \text{ for every } 1 \leq i \leq \ell, \\
&\quad\ \text{we have that } \ell \leq \Delta/2. \\
&< C'(\Delta)
\end{aligned}$$

Hence, there exists a feasible color for v. Since this is true for every iteration, $C^{'}(\Delta)$ colors suffice to acyclically color G. During each iteration as we are examining the neighbors of v up to distance two, it takes $O(\Delta^2)$ time for each iteration. As there are n iterations it takes $O(n\Delta^2)$ for entire graph. □

Notice that the above result, while having a simple proof is slightly wasteful in number of colors. To prove the original result (Theorem 1.6), we investigate the colors in the 3-neighborhood of v and recolor some of the neighbors of v. This helps us to improve the bound on the number of possible dangerous cycles involving v and its neighbors.

2.1 Proof of Lemma 1.4

If each non-single neighbor of v contains at most $\frac{3\Delta}{4}$ colors then we prove that there exists a feasible color α such that:

- it is different from the colors in the $N(v)$ so that the coloring is proper, and
- it doesn't form any dangerous cycles involving path $v_i - v - v_j$ so as to maintain the acyclicity of the coloring.

By borrowing the notation used in the proof of Theorem 2.1, the number of colors that appear exactly once in neighborhood of v is $\Delta - \sum_{i=1}^{\ell} n_i$.

Consider two vertices v_i, v_j, where $1 \leq i, j \leq \Delta$, such that v_i and v_j are colored with the same color. Then, the number of types of possible $v_i - v - v_j$ dangerous cycles is at most $\lfloor \frac{3\Delta}{4} \rfloor$. Extending over all such like colored neighbors of v, we get that there are at most $\sum_{i=1}^{\ell} \lfloor \frac{n_i \lfloor \frac{3\Delta}{4} \rfloor}{2} \rfloor$ types of possible dangerous cycles possible. Hence the number of infeasible colors for v is at most:

$$\underbrace{\sum_{i=1}^{\ell} \lfloor \frac{n_i \lfloor \frac{3\Delta}{4} \rfloor}{2} \rfloor}_{\text{number of types of possible dangerous cycles through } v.} \quad + \quad \underbrace{\Delta - \sum_{i=1}^{\ell} n_i + \ell}_{\text{number of colors in neighborhood of } v.}$$

The above result can be simplified as follows.

$$\begin{aligned}
\sum_{i=1}^{\ell} \lfloor \frac{n_i \lfloor \frac{3\Delta}{4} \rfloor}{2} \rfloor + \Delta - \sum_{i=1}^{\ell} n_i + \ell &\leq \sum_{i=1}^{\ell} \frac{n_i(3\Delta)}{8} + \Delta - \sum_{i=1}^{\ell} n_i + \ell \\
&= \left(\frac{3\Delta}{8} - 1 \right) \sum_{i=1}^{\ell} n_i + \Delta + \ell \\
&\leq \left(\frac{3\Delta}{8} - 1 \right) \Delta + \Delta + \ell, \text{ as } \sum_{i=1}^{\ell} n_i \leq \Delta, \\
&\leq \left(\frac{3\Delta}{8} - 1 \right) \Delta + \Delta + \Delta/2 \text{ as } \ell \leq \Delta/2,
\end{aligned}$$

$$= \frac{3\Delta^2}{8} + \frac{\Delta}{2} < C(\Delta)$$

As the maximum number of infeasible colors is $\frac{3\Delta^2}{8} + \frac{\Delta}{2}$ which is less than $C(\Delta)$ there exists a feasible color α for v. Moreover, finding such a feasible color can be done in $O(\Delta^2)$ time.

2.2 Proof of Lemma 1.5

In this section, we prove Lemma 1.5. The basic idea of the proof is to recolor all the neighbors of v that have more than $3\Delta/4$ colors appearing in their neighborhood. We show that such a recoloring is possible while using only $C(\Delta)$ colors.

Recall that $v_1, v_2, \cdots, v_\Delta$ refer to the neighbors of v. Consider a neighbor v_i of v such that the neighborhood of v_i has more than $3\Delta/4$ colors. Let the neighbors of v_i be $w_1, w_2, \cdots, w_{\Delta-1}$. (Notice that v is a neighbor of v_i, so there are only $\Delta - 1$ neighbors apart from v).

To recolor v_i so that the resulting coloring is still a partial coloring, we have to ensure that v_i does not participate in any dangerous cycles. Additionally, we have to ensure that v_i will become a single vertex after recoloring.

We now arrive at an upper bound on the number of possible dangerous cycles of the form $w_j - v_i - w_k$ for $1 \leq j, k \leq \Delta - 1$. To this end, recall that single vertices do not participate in dangerous cycles. Hence, let m_1 and m_2 be the number of single and non-single vertices in the neighborhood of v_i. We have that $m_1 + m_2 = \Delta - 1$. Further, the number of possible dangerous cycles involving $w_j - v_i - w_k$ for a given $1 \leq j, k \leq \Delta - 1$ is at most $(\Delta - 1)/2$. Over all possible j, k the number of possible dangerous cycles through v_i is at most $\sum_{r=1}^{m_2} \lfloor \frac{m_2(\Delta-1)}{2} \rfloor$. Given that there are at least $3\Delta/4$ colors in the neighborhood of v_i, we argue that $m_2 \leq \lfloor \Delta/2 \rfloor - 2$.

Therefore, the number of colors that appear in the neighborhood of v_i is at most $m_1 + (m_2/2)$, which can be simplified as follows.

$$m_1 + \frac{m_2}{2} = \Delta - 1 - m_2 + \frac{m_2}{2}$$

Since we require that $m_1 + (m_2/2)$ is at least $3\Delta/4$, we require that $m_2 \leq \frac{\Delta}{2} - 2$.

Hence, the number of types of possible dangerous cycles involving v_i is upper bounded by $\left\lfloor \frac{(\lfloor \frac{\Delta}{2} \rfloor - 2)(\Delta - 1)}{2} \right\rfloor$. The new color that we assign for v_i should be distinct from all $\Delta - 1$ colored neighbors of v_i to ensure that the coloring is proper, and it should be distinct from all Δ neighbors of v as v_i has to remain a single vertex. Thus, the number of colors that are not feasible for v is at most:

$$\left\lfloor \frac{(\lfloor \frac{\Delta}{2} \rfloor - 2)(\Delta - 1)}{2} \right\rfloor + \Delta + \Delta - 1 \leq \frac{3\Delta^2 + 4\Delta + 8}{8}.$$

Since the above quantity is less than $C(\Delta)$, we are assured of a feasible color for v_i. So, we define a partial coloring π' as $\pi'(v_i)$ = a feasible color for v_i, and π' agrees with π at every other colored vertex. Such a feasible color can be found in $O(\Delta^3)$ time as we consider the colors in the 3-neighborhood of v.

Under this definition of π', notice that now a feasible color exists for v as follows. All the neighbors of v are either single vertices or have at most $3\Delta/4$ colors in their neighborhood. So, appealing to the proof of Lemma 1.4, a feasible color for v exists within the $C(\Delta)$ colors. This completes the proof of Lemma 1.5.

2.3 To the Main Theorem

To prove Theorem 1.6, we just have to combine the proofs of Lemmata 1.4 and 1.5. Thus, our algorithm to acyclically color a graph G of degree Δ is as follows.

Algorithm AcyclicColor(G)

1. **for** each uncolored vertex v do

    2. **for** each non-single neighbor w of v such that w has at least $\frac{3\Delta}{4} + 1$ colors that appear in its neighborhood do

        3. Recolor w so that w becomes a single vertex

    end-for

    4. Find a feasible color for v within the $C(\Delta)$ colors

end-for

End-Algorithm

Since each iteration of the algorithm takes $O(\Delta^3)$ time, the overall algorithm takes $O(n\Delta^3)$ time. In the above algorithm, Step 3 is facilitated by Lemma 4 and Step 4 is facilitated by Lemma 3.

3 Conclusions

In this paper, we have presented a polynomial time algorithm to acyclically color the vertices of graphs whose maximum degree is Δ using $C(\Delta)$ colors. The algorithm improves the state-of-the-art by a factor of 4/3 by a careful consideration of the colors in a 3-neighborhood.

In future, we wish to extend our result to study the effect of considering the k-neighborhood of a vertex on the number of colors required for an acyclic coloring.

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