Abstract

A vertex magic total labeling of a graph $G = (V, E)$ is a bijection $f : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ such that for every vertex $w$, the sum $f(w) + \sum_{uw \in E} f(uw)$ is a constant. It is well known that all complete graphs $K_n$ admit a vertex magic total labeling. In this paper we present a new proof of this theorem using the concepts of twin factorization and magic square.

Keywords: complete graphs, vertex magic total labeling, factorization.

2000 Mathematics Subject Classification: 05C78 – Graph labelling.

1. Introduction

Let $G = (V, E)$ be a finite, simple, and undirected. For graph theoretic terminology we refer to West [11]. We note a $m = |E|$ and $n = |V|$. The labeling of a graph is a map that takes graph elements $V$ or $E$ or $V \cup E$ to numbers (usually positive or non-negative integers). A comprehensive survey of graph labellings is given in Gallian [1]. The notion of vertex magic total labelling was introduced by Gray et al. [3, 4].

Miller, Macdougall, Slamin, and Wallis [7] gave a vertex magic total labeling of complete graphs $K_n$ for $n \equiv 2 \pmod{4}$ by making use of vertex magic total labeling of $K_2$, where $n/2$ is odd. Lin and Miller [6] gave a vertex magic total labeling of complete graphs $K_n$ for $n \equiv 4 \pmod{8}$ by making use of vertex magic total labeling of $K_4$, where $n/4$ is odd. Labellings for $K_n$, $n$ odd, are presented by Macdougall, Miller, Slamin, and Wallis [8].

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A simpler proof of the fact that complete graphs have a vertex magic total labeling was given by Gray et. al [4].

Focusing on $K_n$, $n$ odd, McQuillan and Smith [9], presented a new technique to arrive at a vertex magic total labeling of complete graphs of odd order for all values of $k$ between $\frac{n(n^2+3)}{4}$ and $\frac{n(n^2+1)}{4}$. Our technique for $K_n$, $n$ odd can be described by using their techniques.

More recently, Gomez [2] has shown that $K_n$ has a super vertex magic labeling, a vertex magic labeling where the smallest labels are given to the vertices. A recent result of Gray shows that all regular graphs have a vertex magic total labeling [3].

Thus, it is clearly established that vertex magic total labeling exists for all complete graphs. In this paper we show, by using new techniques, that every complete graph $K_n$, $n \geq 1$, has a vertex magic total labeling. While our result is not new, the techniques are new and may be of independent interest. Our constructions are very simple, compared to the ones proposed in the literature [6] which uses mutually orthogonal Latin squares. Moreover, while our method is similar to that of [4] in some respects, the magic constant obtained by our method is smaller than the constant reported in [6, 4]. Further, the intuitive representation scheme makes the proposed method easy to understand.

We use ideas from twin-factorizations [10] and magic squares (cf. [5]) to build the desired labellings. In this direction, we first build a vertex magic total labeling of $K_n$ when $n$ is odd (cf. Theorem 2.1). This is then used to build a vertex magic total labeling of $K_n$ when $n$ is even by treating it in three cases: $n \equiv 2 \pmod{4}$, $n \equiv 4 \pmod{8}$, and $n \equiv 0 \pmod{8}$. The labeling for even $n$ is obtained by representing $K_n$ as a union of three graphs: $2K_{n/2}$’s and a $K_{n/2,n/2}$. However, care must be taken when constructing the labels for the subgraphs of the original graph. This requires that (a) the set of labels 1 through $n + n(n - 1)/2$ be partitioned, (b) the resulting partitions be used to label the three subgraphs independently, and (c) combine the labels of the subgraphs to obtain a vertex magic total labeling of the original graph, $K_n$.

1.1. A Note on Representation

To help us in visualizing the labels of the edges and the vertices, we use the matrix representation given in Figure 1 for the labels. We represent the vertex magic total labeling for $K_n$ as an $n \times n$ matrix in which the entries of the first row were used to label the vertices of the graph. The remaining entries of the matrix were used to label the edges of the graph. The $i$th column of the matrix contains the labels of all the edges incident at vertex $i$, in rows 2 through $n$ in the order $(v_i, v_1), (v_i, v_2), \ldots, (v_i, v_{i-1}), (v_i, v_{i+1}), \ldots, (v_i, v_m)$. Notice that in this matrix, each column sum shall be constant, called the magic constant.

The rest of the paper is organized as follows. In Section 2, we show the construction of the labeling for $K_n$, when $n$ is odd. Section 3 shows the construction for the case when $n$ is even. The paper ends with some concluding remarks in Section 4.
2. Vertex magic total labeling for $K_n$ where $n$ is odd

Our construction of vertex magic total labeling of complete graphs $K_n$, where $n$ is odd, is almost similar to construction of a magic square of order $n$. The construction of a magic square of order $n$ is described in [5]. We however have to proceed differently as we do not use all the $n^2$ numbers as is the case with a magic square of order $n$. The details of our scheme are in the following theorem.

Theorem 2.1. $K_n$, $n$ odd, admits a vertex magic total labeling.

Proof. It is trivial for $n = 1$. For $n > 1$, we construct the labeling as follows.

Consider an $n \times n$ matrix $M$. Let $i, j$ be the indices for rows and columns respectively. The indices follow a zero based index. The process of filling the entries of the matrix is similar to constructing a magic square of order $n$. The only difference is that we stop filling the matrix once we have reached the number $n + \frac{n(n-1)}{2}$. The process is described below.

- **Step 1:** Let $x = 1$. Start populating the matrix from the position $j = \frac{(n-1)}{2}$, $i = \frac{(n+1)}{2}$.

- **Step 2:** Fill the current $(i, j)$ position with the value $x$, and increment $x$ by 1.

- **Step 3:** The subsequent entries are filled by moving south-east by one position (i.e. $i = (i+1) \mod n, j = (j+1) \mod n$), till a vacant entry is found. If the cell is already filled, then fill from the south-west direction ($i = i + 1 \mod n, j = j - 1 \mod n$). The value of $x$ has to be incremented after filling an entry. If $x$ reaches $n + \frac{n(n-1)}{2}$, then proceed to Step 4.

- **Step 4:** The remaining $n(n-1)/2$ entries are filled just by copying the non-diagonal entries by appealing to symmetry (i.e., just copying the numbers in the lower triangular matrix to the upper triangular matrix and vice-versa from filled cells to unfilled cells).
• **Step 5:** Rearrange the entries of the matrix so that the numbers placed along the principal diagonal are moved to the first row of the matrix and all the entries in the upper triangular portion of the matrix are moved down by one position.

**Lemma 2.2.** The above labeling is a valid vertex total magic labeling of $K_n$, $n$ odd. Moreover, the magic constant obtained is $S(n) = \frac{n^3 + 3n^2}{4n}$.

**Proof.** Notice that the only difference from a magic square to our labeling matrix is that while in a magic square the entries range from 1 to $n^2$, here we have used only labels from 1 to $n + n(n - 1)/2$. The other entries can be seen to be reduced in magnitude by an amount of $n(n - 1)/2$. Since this applies to all the entries, it can be seen that the properties of the magic square that all column sums are equal still applies in our case. Hence, the labeling obtained is a vertex magic total labeling of $K_n$, where $n$ is odd.

It can be seen that the magic constant using the above construction is $S(n) = \frac{n^3 + 3n^2}{4}$.

**Example 2.3.** We provide an example below with $n = 5$.

The matrices in Figure 2 show the result obtained after completion of Step 3 and Step 5.

<table>
<thead>
<tr>
<th></th>
<th>11</th>
<th>07</th>
<th>03</th>
</tr>
</thead>
<tbody>
<tr>
<td>04</td>
<td>12</td>
<td>08</td>
<td></td>
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<tr>
<td>05</td>
<td>13</td>
<td>09</td>
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<tr>
<td>10</td>
<td>01</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>06</td>
<td>02</td>
<td>15</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Vertex label</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incident edges</td>
<td>04</td>
<td>04</td>
<td>07</td>
<td>10</td>
<td>03</td>
</tr>
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<td></td>
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<td>09</td>
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<tr>
<td></td>
<td>03</td>
<td>06</td>
<td>09</td>
<td>02</td>
<td>02</td>
</tr>
</tbody>
</table>

Figure 2: The resulting matrices after Step 3, on the left, and Step 5, on the right.

3. **Vertex Magic Total Labeling for $K_N$, $N$ even**

In this section, we address the construction of vertex magic total labeling for $K_n$, when $n$ is even. This is done in three steps. We treat the case where $n \equiv 2 \mod 4$ (cf. Theorem 3.1) first as in that case $n/2$ is odd. This lets us use the result of Theorem 2.1 and compose labellings of smaller graphs to arrive at the labeling of $K_n$. This is then extended to the case where $n \equiv 4 \mod 8$ and then finally where $n \equiv 0 \mod 8$.

In the remainder of this section, the case of $n \equiv 2 \mod 4$ is treated in Section 3.1, the case of $n \equiv 4 \mod 8$ in Section 3.2, and the case of $n \equiv 0 \mod 8$ is presented in Section 3.3.
3.1. Vertex Magic Total Labeling for \( K_N, N \equiv 2 \pmod{4} \)

**Theorem 3.1.** There is a vertex magic total labeling for \( K_N \), for all \( N \equiv 2 \pmod{4} \).

**Proof.** Let \( n = N/2 \). Our construction of vertex magic labeling for \( K_N \) makes use of a magic square of order \( n \) and a vertex magic total labeling of \( K_n \). The construction of a magic square of order \( n \) and vertex magic total labelling for \( K_n \) is described in Section 2.

The basic idea behind constructing a vertex magic total labeling of these complete graphs is to view the graph \( K_N \) as a union of two complete graphs each of \( n \) vertices and a complete bipartite graph with \( n \) vertices on each side, i.e., \( K_N = K_n \cup K_n \cup K_{n,n} \). We use the numbers 1 through \( N+N(N-1)/2 \) for the labels of the vertices and edges of \( K_N \).

To arrive at the labeling, we partition this set of numbers into five disjoint sets as follows.

\[
\begin{align*}
S_1 & = \bigcup_{i=1}^{n-1/2} \{(2i+1)n + 1, (2i+1)n+2, \ldots, (2i+2)n\} \cup \{1, 2, \ldots, n\} \\
S_2 & = \bigcup_{i=2}^{n-1/2} \{2i \cdot n + 1, 2i \cdot n + 2, \ldots, (2i+1)n\} \cup \{n+1, n+2, \ldots, 2n\} \\
S_3 & = \{2n + 1, 2n + 2, \ldots, 3n\}, \\
S_4 & = \{n^2 + n + 1, n^2 + n + 2, \ldots, n^2 + 2n\}, \\
S_5 & = \{n^2 + 1, n^2 + 2, \ldots, n^2 + n\} \cup \{n^2 + 2n + 1, n^2 + 2n + 2, \ldots, 2n^2 + n\}.
\end{align*}
\]

In our construction, we use sets \( S_1 \) and \( S_4 \) for constructing the intermediate labeling \( L_1 \) for \( K_{n/2} \). The elements of \( S_1 \) are used to label the edges and elements of \( S_4 \) are used to label the vertices. Similarly, we use sets \( S_2 \) and \( S_3 \) for constructing the intermediate labeling \( L_2 \) for another \( K_{n/2} \). The elements of \( S_2 \) are used to label the edges and elements of \( S_3 \) are used to label the vertices. The elements of \( S_5 \) are used to label the edges in the bipartite graph \( K_{n/2,n/2} \). An illustration of the construction is shown in Figure 3.

![Figure 3: The construction of the labeling for \( K_N \), with \( N \equiv 2 \pmod{4} \). The two \( K_{n/2} \)s and \( K_{n/2,n/2} \) are shown along with the sets that are used to label the various components of \( K_n \) are also shown in the picture.](image-url)
For constructing $L_1$ and $L_2$, first consider a vertex magic total labelling $L$ for $K_n$. Since $n$ is odd, this can be obtained using the construction given in Section 2. The matrices $L_1$ and $L_2$ are obtained from $L$ as follows. Replace the vertex labels of $L$ by the elements in $S_4$ and replace the edge labels by the elements of $S_1$, to get $L_1$. Similarly, if the edge labels are replaced by elements of $S_2$ and the vertex labels by $S_3$, we get $L_2$.

For constructing the labels of the edges in the bipartite graph $K_{n/2,n/2}$, consider a magic square of order $n/2$ and replace the elements of the magic square by the elements of $S_5$. Let us name this modified magic square as $N_1$. Transpose the magic square $N_1$ and name this as $N_2$. Finally, arrange all these intermediate labellings as shown below to get the labeling for $K_N$.

$$
\begin{bmatrix}
L_1 & L_2 \\
N_1 & N_2
\end{bmatrix}
$$

It can be deduced that the magic constant in this case is $8n^3 + 6n^2 - 2n$.

Notice, however, that the above arrangement of labels needs to be adjusted to get the labeling matrix in the standard form described in Section 1.1. This rearrangement is required as the the first column of $L_2$ corresponds to edge labels $v_{n+1}v_{n+2}, v_{n+1}v_{n+3}, \ldots, v_{n+1}v_{2n}$ and the first column of $N_2$ corresponds to the edge labels of $v_{n+1}v_1, v_{n+1}v_2, \ldots, v_{n+1}v_n$. To get to the standard form, we need to shift the rows of $N_2$ by $n - 1$ places up and shift the rows 2 through $n - 1$ of $L_2$ by $n$ places.

**Example 3.2.** Let us construct vertex magic total labelling for $K_6$ using a vertex magic total labelling of $K_3$. Let $L$ below be the vertex magic total labeling of $K_3$, obtained by using the procedure of Section 2.

<table>
<thead>
<tr>
<th>vertex label</th>
<th>04</th>
<th>05</th>
<th>06</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incident edges</td>
<td>03</td>
<td>03</td>
<td>02</td>
</tr>
<tr>
<td></td>
<td>02</td>
<td>01</td>
<td>01</td>
</tr>
</tbody>
</table>

From this $L$, $L_1$ is obtained by using elements of $S_1$ and $S_4$, with $S_1 = \{1, 2, 3\}$, and $S_4 = \{13, 14, 15\}$, we get the following matrix as $L_1$. Similarly, $L_2$ is obtained by using elements of $S_2$ and $S_4$ with $S_2 = \{4, 5, 6\}$ and $S_3 = \{7, 8, 9\}$.

$$
L_1 = \begin{bmatrix}
13 & 14 & 15 \\
03 & 03 & 02 \\
02 & 01 & 01
\end{bmatrix}
$$

$$
L_2 = \begin{bmatrix}
07 & 08 & 09 \\
06 & 06 & 05 \\
05 & 04 & 04
\end{bmatrix}
$$

To construct intermediate labelling $N_1$ and $N_2$, consider a magic square of order three as follows. In Figure 4, the matrix on the left shows a magic square of order three, the matrix on the right shows the magic square with elements replaced from elements in $S_5$. 
Figure 4: The construction of $N_1$ from a magic square of order 3

The matrix $N_2$, for reasons of symmetry, is constructed as the transpose of $N_1$. The final matrix that shows a vertex magic total labeling of $K_6$, obtained by putting together $L_1, L_2, N_1$, and $N_2$, after the required rearrangement is shown in Figure 5.

<table>
<thead>
<tr>
<th>Vertex label</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incident Edges</td>
<td>03</td>
<td>02</td>
<td>02</td>
<td>16</td>
<td>12</td>
<td>20</td>
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<td>10</td>
<td>18</td>
<td>05</td>
<td>04</td>
<td>04</td>
</tr>
</tbody>
</table>

Figure 5: A vertex magic total labeling of $K_6$. The first row elements are the vertex labels. The column entries of the $i$th column give the edge labels of edges adjacent to vertex $v_i$. See also the representation in Figure 1.

3.2. Labeling for $K_N$ where $N \equiv 4 \mod 8$

We now move to the case where $N \equiv 4 \mod 8$. We show the following theorem.

**Theorem 3.3.** There is a vertex magic total labeling for $K_N$, for all $N \equiv 4 \mod 8$.

**Proof.** Our construction of vertex magic labeling for $K_N$ makes use of magic squares of order $N/4$. Let $n = N/4$. The construction of a magic square of order $n$ and a vertex magic total labelling for $K_n$ is known to exist from Section 1. The basic idea behind constructing a vertex magic total labeling of $K_N$, $N \equiv 4 \mod 8$, is to represent $K_N$ as the union of four $K_n$’s, two $K_n, n$s, and a $K_{2n,2n}$.

It is required to map the vertices and edges in these smaller graphs to numbers from 1 through $N + N(N - 1)/2$. To this end, we partition the integers from 1 through $N + N(N - 1)/2$ into eleven disjoint sets $S_1$ through $S_{11}$ as follows.

- $S_1 = \cup_{i=1}^{n-1/2} \{(2i + 1)n + 1, (2i + 1)n + 2, \ldots, (2i + 2)n\} \cup \{1, 2, \ldots, n\}$
- $S_2 = \cup_{i=2}^{n-2/2} \{2i \cdot n + 1, 2i \cdot n + 2, \ldots, (2i + 1)n\} \cup \{n + 1, n + 2, \ldots, 2n\}$
- $S_3 = \{2n + 1, 2n + 2, \ldots, 3n\}$
- $S_4 = \{n^2 + n + 1, n^2 + n + 2, \ldots, n^2 + 2n\}$
- $S_5 = \{n^2 + 1, n^2 + 2, \ldots, n^2 + n\} \cup \{n^2 + 2n + 1, n^2 + 2n + 2, \ldots, 2n^2 + n\}$. 
Figure 6: The construction of the labeling for $K_N$, with $N \equiv 4 \mod 8$. The four smaller $K_n$s are labeled I, II, III, and IV. The sets that are used to label the various components of $K_N$ are also shown in the picture. The dashed edges correspond to $K_{2n,2n}$.

The sets $S_6, S_7, S_8, S_9,$ and $S_{10}$ are constructed by adding $2n^2 + n$ to each element of $S_1$ through $S_5$ respectively. Finally, $S_{11} = \{4n^2 + 2n + 1, 4n^2 + 2n + 2, \ldots , 8n^2 + 2n\}$ will be a set of $4n^2$ elements.

The way we use these sets is as follows. We use sets $S_1$ and $S_4$ for constructing intermediate labeling $L_1$ for a $K_n$ with the elements of $S_1$ used to label the edges and elements of $S_4$ for labeling the vertices. Similarly, for labeling another $K_n$, we use the elements of $S_2$ to label the edges and the elements of $S_3$ to label the vertices. For the third $K_n$, we use the elements of $S_6$ to label the edges and the elements of $S_8$ to label the vertices. Similarly, for the fourth $K_n$, we use the elements of $S_7$ to label the edges and the elements of $S_9$ to label the vertices. In the above, the terms first $K_n$ etc. are pictured in Figure 6.

Notice that we have to label the edges between the vertices in the first and the second $K_n$. For this we use the elements of $S_5$. Similarly, the elements of $S_{10}$ are used to label the edges between the third and the fourth $K_n$. We are now left with the edges between the first and second $K_n$ to the third and fourth $K_n$. For this we use the elements of $S_{11}$.

For constructing $L_1$ through $L_4$, we use a similar approach used in the proof of Theorem
3.1. We first construct a magic square of order $n$ and replace the edge and vertex labels with the elements of the appropriate set.

We now focus on the construction of $N_1$ through $N_4$, which serve as edge labels of the edges between the first and second $K_n$ and edges between the third and the fourth $K_n$. Consider a magic square of order $n$ and replace the entries by the elements of $S_5$ to get $N_1$. We define $N_2 = N_1^T$. The entries in $N_1$ and $N_2$ thus serve as the labels of the edges between the first and second $K_n$. Similarly, we obtain $N_3$ by replacing the entries of a magic square of order $n$ by the entries of $S_{10}$. We then define $N_4 = N_3^T$. The entries in $N_3$ and $N_4$ thus serve as the labels of the edges between the third and fourth $K_n$.

Finally, we show the construction of labels for the edges from the first and second $K_n$ to the third and the fourth $K_n$. These labels are from the set $S_{11}$ which has $4n^2$ numbers. We form these labels as four matrices $M_1$ through $M_4$ and their transposes. The matrix $M_i$, for $1 \leq i \leq 4$, is obtained by replacing the entries of a magic square of order $n$ with the entries of $S_{11}$ ranging from $(i - 1)n^2 + 1$ to $i \cdot n^2$.

Finally, arrange all these intermediate labellings as shown below.

\[
\begin{bmatrix}
L_1 & L_2 & L_3 & L_4 \\
N_3 & N_4 & N_1 & N_2 \\
M_1 & M_2 & M_1^T & M_2^T \\
M_4 & M_3 & M_4^T & M_3^T
\end{bmatrix}
\]

It can be deduced that the magic constant in this case is $\frac{32n^3+13n^2+n}{2}$.

As in Section 3.1, notice that a rearrangement of labels as shown below has to be done to arrive at the standard form given in Section 1.1. In the table below, the notation $M : a$ refers to the $a$th row of the matrix $M$ for a positive integer $a$, and the notation $M : a \cdot \cdot b$ refers to the rows $a$ through $b$ (both inclusive) of the matrix $M$, for positive integers $a, b$, $a < b$.

\[
\begin{bmatrix}
L_1 : 1 & L_2 : 1 & L_3 : 1 & L_4 : 1 \\
L_1 : 2 \cdot \cdot n & N_4 : 1 \cdot \cdot n - 1 & M_1^T : 1 \cdot \cdot n - 1 & M_4^T : 1 \cdot \cdot n - 1 \\
N_3 : 1 & N_4 : n & M_1^T : n & M_4^T : n \\
N_3 : 2 \cdot \cdot n & L_2 : 2 \cdot \cdot n & M_1^T : 1 \cdot \cdot n - 1 & M_4^T : 1 \cdot \cdot n - 1 \\
M_1 : 1 & M_2 : 1 & M_1^T : n & M_4^T : n \\
M_1 : 2 \cdot \cdot n & M_3 : 2 \cdot \cdot n & N_1 : 2 \cdot \cdot n & L_4 : 2 \cdot \cdot n
\end{bmatrix}
\]

**Example 3.4.** Let us construct vertex magic total labelling for $K_{12}$ using a vertex magic total labeling of $K_3$. 

Let $L$, shown below, be the vertex magic total labeling of $K_3$ obtained from using the approach of Section 2.

<table>
<thead>
<tr>
<th>vertex label</th>
<th>04</th>
<th>05</th>
<th>06</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incident edges</td>
<td>03</td>
<td>03</td>
<td>02</td>
</tr>
<tr>
<td></td>
<td>02</td>
<td>01</td>
<td>01</td>
</tr>
</tbody>
</table>

From this, the matrix $L_1$ is obtained by using $S_1 = \{1, 2, 3\}$ for the edge labels and $S_4 = \{13, 14, 15\}$ for the vertex labels. The resulting $L_1$ is shown below.

<table>
<thead>
<tr>
<th>Vertex label</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incident edges</td>
<td>03</td>
<td>03</td>
<td>02</td>
</tr>
<tr>
<td></td>
<td>02</td>
<td>01</td>
<td>01</td>
</tr>
</tbody>
</table>

Using $S_2 = \{4, 5, 6\}$, $S_3 = \{7, 8, 9\}$, $S_6 = \{22, 23, 24\}$, and $S_9 = \{34, 25, 36\}$, we obtain the matrices $L_2$, $L_3$, and $L_4$ respectively as described in the proof of Theorem 3.3. The resulting matrices are shown below.

\[
L_2 = \begin{bmatrix}
7 & 8 & 9 \\
6 & 6 & 5 \\
5 & 4 & 4
\end{bmatrix} \quad L_3 = \begin{bmatrix}
34 & 35 & 36 \\
24 & 24 & 23 \\
23 & 22 & 22
\end{bmatrix} \quad L_4 = \begin{bmatrix}
28 & 29 & 30 \\
27 & 27 & 26 \\
26 & 25 & 25
\end{bmatrix}
\]

To construct the matrices $N_1$ and $N_2$, consider a magic square of order three and the set $S_5 = \{10, 11, 12, 16, 17, 18, 19, 20, 21\}$. Replace the elements of the magic square by elements of $S_5$ to get $N_1$. The resulting $N_1$ and $N_2$ are shown below.

\[
N_1 = \begin{bmatrix}
16 & 21 & 11 \\
12 & 17 & 19 \\
20 & 10 & 18
\end{bmatrix} \quad N_2 = \begin{bmatrix}
16 & 12 & 20 \\
21 & 17 & 10 \\
11 & 19 & 18
\end{bmatrix}
\]

Similarly, using the elements of the sets $S_{10} = \{31, 32, 33, 37, 38, 39, 40, 41, 42\}$, the matrices $N_3$ and $N_4$ are obtained as below.

\[
N_3 = \begin{bmatrix}
37 & 42 & 32 \\
33 & 38 & 40 \\
41 & 31 & 39
\end{bmatrix} \quad N_4 = \begin{bmatrix}
37 & 33 & 41 \\
42 & 38 & 31 \\
32 & 40 & 39
\end{bmatrix}
\]

Finally, the set $S_{11}$ has 36 consecutive elements from 43 to 78. Of these, the first nine elements are used to construct the matrices $M_1$, the second nine elements are used to construct $M_2$, the third nine elements for $M_3$ and the final nine elements for $M_4$. For purposes of brevity, we show only $M_1$ and $M_2$ in the following. The matrix $M_1$ is obtained as a magic square of order three with elements from 43 to 51.
The final matrix after combining all of these matrices is shown in Figure 7.

<table>
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<tr>
<th>Vertex label</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>34</th>
<th>35</th>
<th>36</th>
<th>28</th>
<th>29</th>
<th>30</th>
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<tbody>
<tr>
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<td>6</td>
<td>6</td>
<td>5</td>
<td>24</td>
<td>24</td>
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<td>76</td>
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</tbody>
</table>

Figure 7: A vertex magic total labeling of $K_{12}$. As earlier, the elements of the first row are the vertex labels and the remaining column entries are the edge labels.

The above matrix elements have to be rearranged to get the label matrix in the standard form.

3.3. Labeling for $N \equiv 0 \pmod{8}$

Finally, we focus on the case where $n \equiv 0 \pmod{8}$. In this case, we use mostly similar techniques as that of Theorem 3.1. Hence, we provide an outline of the proof for the following theorem.

**Theorem 3.5.** There is a vertex magic total labeling for $K_N$, for all $N \equiv 0 \pmod{8}$.

**Proof.** (Outline). Notice that in this case, we can represent $K_N$ as two $K_{N/2}$ and a $K_{N/2,N/2}$ with $N/2 \equiv 0 \pmod{4}$. We then use the constructions of Theorem 3.3 and Theorem 3.1 to combine the labels of the $K_{N/2}$ with the edge labels of $K_{N/2,N/2}$ and arrive at a vertex magic total labeling of $K_N$ with $N \equiv 0 \pmod{8}$.

4. Conclusion

In this paper, we used cohesive techniques to show that every complete graph has a vertex magic total labeling. Our techniques attempt to draw connections between graph
factorization and graph labeling. It is worth to explore the stronger connections between the two.

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References


